# Factorization in Haar system Hardy spaces Workshop in Analysis and Probability Seminar

#### Thomas Speckhofer

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July 23, 2024

### Overview

Introduction

Definitions

- Main results
- Proofs

#### **Primary Banach spaces**

- ullet Let X be a Banach space.
- Let  $\mathcal{B}(X)$  be the set of all bounded linear operators  $T\colon X\to X$ .
- X is called *primary* if for all spaces Y,Z, we have that  $X\sim Y\oplus Z$  implies  $Y\sim X$  or  $Z\sim X$ .
- Examples:  $c_0$ ,  $\ell^p$ ,  $L^p$   $(1 \le p \le \infty)$ ,  $H^1$ , some rearrangement-invariant function spaces, ...

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- Let  $P \in \mathcal{B}(X)$  be a projection. **Goal**: Show that P(X) or  $(I_X P)(X)$  has a complemented subspace isomorphic to X. (\*)
- If X satisfies (\*) and  $X \sim \ell^p(X)$  for some  $1 \leq p \leq \infty$ , then by Pełczyński's decomposition method, X is primary.
- lacktriangle Sufficient for (\*): X has the primary factorization property.

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$$\mathcal{M}_X = \{ T \in \mathcal{B}(X) : I_X \text{ does not factor through } T \}.$$

- The set  $\mathcal{M}_X$  is an ideal of  $\mathcal{B}(X) \iff X$  has the primary factorization property (see Dosev-Johnson [1]).
- In that case,  $\mathcal{M}_X$  is the unique maximal ideal of  $\mathcal{B}(X)$ .

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- Dyadic intervals:  $\mathcal{D} = \left\{ [0,1), [0,\frac{1}{2}), [\frac{1}{2},1), [0,\frac{1}{4}), [\frac{1}{4},\frac{1}{2}), \dots \right\}$
- $I^+ = \text{left half}, I^- = \text{right half of } I \in \mathcal{I}$
- Define  $h_I=\mathbb{1}_{I^+}-\mathbb{1}_{I^-}$ ,  $I\in\mathcal{D}$ .
- Put  $h_{\varnothing} = \mathbb{1}_{[0,1)}$  and  $\mathcal{D}^+ = \mathcal{D} \cup \{\varnothing\}$ .
- The Haar system  $(h_I)_{I\in\mathcal{D}^+}$  is a *Schauder basis* for  $L^p$ ,  $1\leq p<\infty$
- $D \in \mathcal{B}(L^p)$  is called a *Haar multiplier* if  $Dh_I = d_I h_I$  for all  $I \in \mathcal{D}^+$   $(d_I \in \mathbb{R}, I \in \mathcal{D})$ .

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### Theorem (R. Lechner and T. S. '23)

Suppose that  $\|\cdot\|_Y \nsim \|\cdot\|_{L^{\infty}}$  on the dyadic simple functions. Let E be one of the following spaces:

- (i) E=Y
- (ii)  $E = \ell^p(Y)$  for some  $1 \le p < \infty$
- (iii)  $E = \ell^{\infty}(Y)$  if Y is "asymptotically curved" w.r.t.  $(h_I)_{I \in \mathcal{D}}$ .

Then E has the primary factorization property, and hence,  $\mathcal{M}_E$  is the unique maximal ideal of  $\mathcal{B}(E)$ . In particular, the spaces in (ii) and (iii) are primary.

- Basic idea: Step-by-step reduction.

$$D \approx A_1 T B_1, \quad D^{\text{stab}} = A_2 D B_2, \dots$$

$$\hat{h}_I = \sum_{K \in \mathcal{B}_I} \varepsilon_K h_K, \qquad \mathcal{B}_I \subseteq \mathcal{D}, \ \varepsilon_K = \pm 1$$

### Proof method

- Basic idea: Step-by-step reduction. Operator T o Haar multiplier D o stable Haar multiplier  $D^{\mathrm{stab}}$
- ullet Clearly, the identity factors through  $cI_Y$  or  $(1-c)I_{Y^{\perp}}$

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ullet How are  $A_i, B_i$  defined? o faithful Haar system  $(\hat{h}_I)_{I\in\mathcal{D}}$ 

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 $\bullet$   $\mathcal{B}_I$  are pairwise disjoint and satisfy some compatibility conditions.

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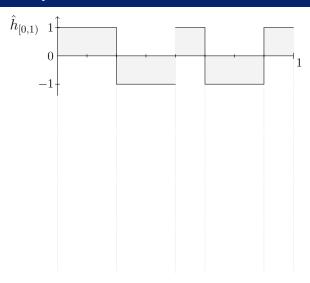
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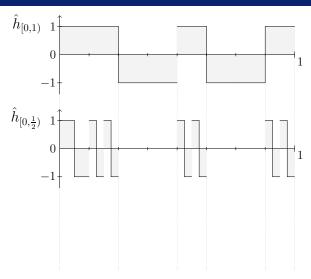
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# Faithful Haar systems



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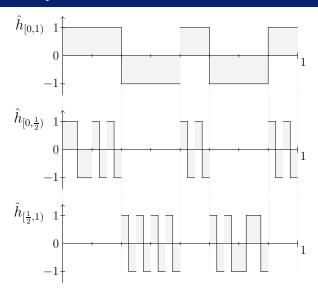
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$$Bx = \sum_{I \in \mathcal{D}} \frac{\langle h_I, x \rangle}{|I|} \hat{h}_I, \qquad Ax = \sum_{I \in \mathcal{D}} \frac{\langle \hat{h}_I, x \rangle}{|I|} h_I$$

#### Diagonalization

• Let  $I, J \in \mathcal{D}$  with "I < J". Given  $\hat{h}_I$ , construct  $\hat{h}_J$  out of sufficiently high-frequency "building blocks"  $h_K$ .

$$\Longrightarrow |\langle \hat{h}_I, T\hat{h}_J 
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#### Stabilization

- Let  $D: Y \to Y$  be a Haar multiplier. Put  $r_n = \sum_{|K|=2^{-n}} h_K$  and  $r_n^{\Gamma} = \mathbb{1}_{\Gamma} \cdot r_n \ (n > 0, \ \Gamma \subset [0, 1)).$

for each dyadic 
$$\Gamma\subseteq[0,1),\quad \left(\langle r_n^\Gamma,Dr_n^\Gamma
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$$\hat{h}_I = \sum_{K \in \mathcal{B}_I} \theta_K h_K, \qquad (\theta_K)_{K \in \mathcal{B}_I} \in \{\pm 1\}^{\mathcal{B}_I}$$

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•  $\Longrightarrow$  The entries  $d_I^{\mathrm{stab}}$  of  $D^{\mathrm{stab}} = ADB$  satisfy

$$\mathbb{E} \, d_{I^{\pm}}^{\mathrm{stab}} \approx d_{I}^{\mathrm{stab}},$$

and the variance is small  $\rightarrow$  choose a "good" realization of  $(\theta_K)$ .

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# Conclusion of the proof

- Perturbation argument  $\implies cI_Y ADB$  is small
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- Perturbation argument  $\implies cI_Y ADB$  is small.
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# Factorization through operators with large diagonal

• An operator  $T: Y \to Y$  has large diagonal (w.r.t. the Haar basis) if

$$\inf_{I\in\mathcal{D}}\frac{|\langle h_I, Th_I\rangle|}{|I|} > 0.$$

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# Theorem (R. Lechner and T. S. '23)

Let Y be a Haar system Hardy space with  $\|\cdot\|_{Y} \not\sim \|\cdot\|_{L^{\infty}}$ . Then the identity  $I_Y$  factors through all operators  $T \in \mathcal{B}(Y)$  with large diagonal, i.e.,  $(h_I)_{I\in\mathcal{D}}$  has the factorization property in Y.

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• Analogous results for  $\ell^p$ -sums of Haar system Hardy spaces.

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- $\bullet$  First step: Switch to large positive diagonal:  $\frac{\langle h_I, Th_I \rangle}{|I|} \geq \delta$  for all I(Gamlen-Gaudet)

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  - Our approach: Stabilization yields  $c \geq \delta$ . Works in all Haar system Hardy spaces.

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# Thank you for your attention!

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