# Factorization in Haar system Hardy spaces

Workshop in Analysis and Probability Seminar

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July 23, 2024

# Overview

# Introduction

## Primary Banach spaces

- Let $X$ be a Banach space.
- Let $\mathcal{B}(X)$ be the set of all bounded linear operators $T: X \to X$.
- $X$ is called *primary* if for all spaces $Y, Z$, we have that $X \sim Y \oplus Z$ implies $Y \sim X$ or $Z \sim X$.
- Examples: $c_0$, $\ell^p$, $L^p$ $(1 \leq p \leq \infty)$, $H^1$, some rearrangement-invariant function spaces, ...

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  some rearrangement-invariant function spaces, ...

# Primariness and factorization

- Let $P \in \mathcal{B}(X)$ be a projection. **Goal**: Show that $P(X)$ or $(I_X - P)(X)$ has a complemented subspace isomorphic to $X$. $(*)$
- If $X$ satisfies $(*)$ and $X \sim \ell^p(X)$ for some $1 \leq p \leq \infty$, then by Pełczyński's decomposition method, $X$ is primary.
- Sufficient for $(*)$: $X$ has the *primary factorization property*.

**Definition**

Let $S, T \in \mathcal{B}(X)$. We say that $S$ *factors through* $T$ if there are $A, B \in \mathcal{B}(X)$ such that $S = ATB$.

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We say that a Banach space $X$ has the *primary factorization property* if for every $T \in \mathcal{B}(X)$, the identity $I_X$ factors through $T$ or $I_X - T$.

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- For a Banach space $X$, define

$$\mathcal{M}_X = \{T \in \mathcal{B}(X) : I_X \text{ does } \textbf{not} \text{ factor through } T\}.$$

- The set $\mathcal{M}_X$ is an ideal of $\mathcal{B}(X) \iff X$ has the primary factorization property (see Dosev-Johnson [1]).
- In that case, $\mathcal{M}_X$ is the *unique maximal ideal* of $\mathcal{B}(X)$.

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# Definitions

- Dyadic intervals: $\mathcal{D} = \{[0,1), [0,\frac{1}{2}), [\frac{1}{2},1), [0,\frac{1}{4}), [\frac{1}{4},\frac{1}{2}), \dots\}$
- $I^+$ = left half, $I^-$ = right half of $I \in \mathcal{D}$
- Define $h_I = \mathbb{1}_{I^+} - \mathbb{1}_{I^-}$, $I \in \mathcal{D}$.
- Put $h_\varnothing = \mathbb{1}_{[0,1)}$ and $\mathcal{D}^+ = \mathcal{D} \cup \{\varnothing\}$.
- The Haar system $(h_I)_{I \in \mathcal{D}^+}$ is a *Schauder basis* for $L^p$, $1 \leq p < \infty$.
- $D \in \mathcal{B}(L^p)$ is called a *Haar multiplier* if $Dh_I = d_I h_I$ for all $I \in \mathcal{D}^+$ ($d_I \in \mathbb{R}$, $I \in \mathcal{D}$).

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# Main results

- A **Haar system space** $X$ is the completion of $H = \operatorname{span}\{h_I : I \in \mathcal{D}^+\}$ under a norm $\|\cdot\|_X$ such that:
  - If $x, y \in H$ and $|x|$, $|y|$ have the same distribution, then $\|x\|_X = \|y\|_X$.
  - $\|\mathbb{1}_{[0,1)}\|_X = 1$.
- Examples: $L^p$, $1 \leq p < \infty$, all separable rearrangement-invariant function spaces
- Let $\mathbf{r} = (r_I)_{I\in\mathcal{D}}$ be a *constant* or *independent* family of random variables uniformly distributed on $\{+1, -1\}$.
- **Haar system Hardy space** $X(\mathbf{r})$: completion of $\operatorname{span}\{h_I\}_{I\in\mathcal{D}}$ under

$$\Big\|\sum_{I\in\mathcal{D}} a_I h_I\Big\|_{X(\mathbf{r})} = \Big\|s \mapsto \mathbb{E}\Big|\sum_{I\in\mathcal{D}} r_I a_I h_I(s)\Big|\Big\|_X$$

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# Main results

- Now fix a Haar system Hardy space $Y$.

**Theorem (R. Lechner and T. S. '23)**

*Suppose that $\|\cdot\|_Y \not\sim \|\cdot\|_{L^\infty}$ on the dyadic simple functions. Let $E$ be one of the following spaces:*

(i) $E = Y$

(ii) $E = \ell^p(Y)$ *for some* $1 \leq p < \infty$

(iii) $E = \ell^\infty(Y)$ *if $Y$ is "asymptotically curved" w.r.t.* $(h_I)_{I\in\mathcal{D}}$.

*Then $E$ has the primary factorization property, and hence, $\mathcal{M}_E$ is the unique maximal ideal of $\mathcal{B}(E)$. In particular, the spaces in* (ii) *and* (iii) *are primary.*

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# Proof method

- Basic idea: Step-by-step reduction.
  Operator $T \to$ Haar multiplier $D \to$ stable Haar multiplier $D^{\mathrm{stab}}$ $\to$ constant multiple of the identity $cI_Y$
- Clearly, the identity factors through $cI_Y$ or $(1-c)I_Y$.

$$D \approx A_1 T B_1, \quad D^{\mathrm{stab}} = A_2 D B_2, \ \ldots$$

- How are $A_i, B_i$ defined? $\to$ *faithful Haar system* $(\hat{h}_I)_{I\in\mathcal{D}}$

$$\hat{h}_I = \sum_{K \in \mathcal{B}_I} \varepsilon_K h_K, \qquad \mathcal{B}_I \subseteq \mathcal{D},\ \varepsilon_K = \pm 1$$

- $\mathcal{B}_I$ are pairwise disjoint and satisfy some compatibility conditions.

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  Operator $T \to$ Haar multiplier $D \to$ stable Haar multiplier $D^{\mathrm{stab}}$ $\to$ constant multiple of the identity $cI_Y$
- Clearly, the identity factors through $cI_Y$ or $(1-c)I_Y$.

$$D \approx A_1 T B_1, \quad D^{\mathrm{stab}} = A_2 D B_2, \ \ldots$$

- How are $A_i, B_i$ defined? $\to$ *faithful Haar system* $(\hat{h}_I)_{I \in \mathcal{D}}$

$$\hat{h}_I = \sum_{K \in \mathcal{B}_I} \varepsilon_K h_K, \qquad \mathcal{B}_I \subseteq \mathcal{D},\ \varepsilon_K = \pm 1$$

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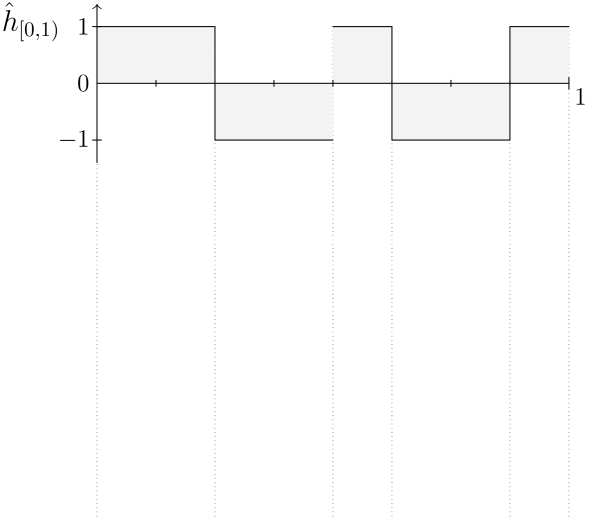

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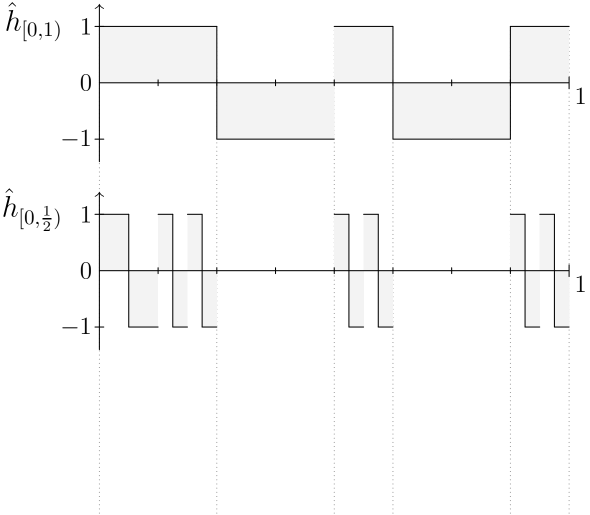

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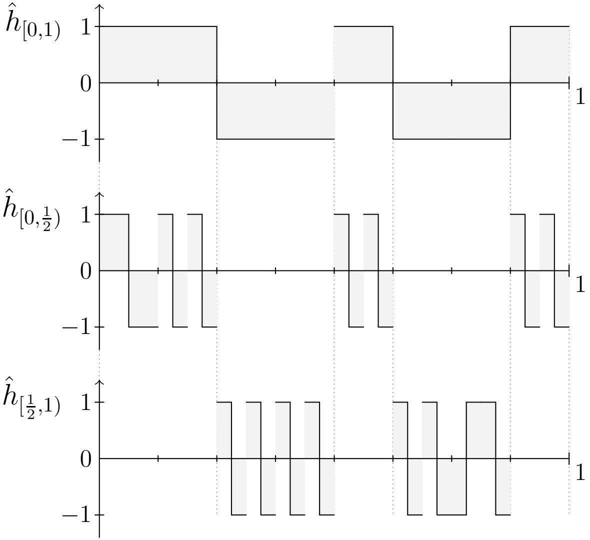

# Proof

- Associated operators $A, B$:

$$Bx = \sum_{I \in \mathcal{D}} \frac{\langle h_I, x \rangle}{|I|} \hat{h}_I, \qquad Ax = \sum_{I \in \mathcal{D}} \frac{\langle \hat{h}_I, x \rangle}{|I|} h_I$$

## Diagonalization

- Let $I, J \in \mathcal{D}$ with "$I < J$". Given $\hat{h}_I$, construct $\hat{h}_J$ out of sufficiently high-frequency "building blocks" $h_K$.

$$\implies |\langle \hat{h}_I, T\hat{h}_J \rangle| \text{ and } |\langle \hat{h}_J, T\hat{h}_I \rangle| \text{ are small.}$$

- Put $Dh_I = \frac{\langle \hat{h}_I, T\hat{h}_I \rangle}{|I|} h_I$, $I \in \mathcal{D}$.

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## Stabilization

- Let $D\colon Y \to Y$ be a Haar multiplier. Put $r_n = \sum_{|K|=2^{-n}} h_K$ and $r_n^\Gamma = \mathbb{1}_\Gamma \cdot r_n$ ($n \geq 0$, $\Gamma \subset [0,1)$).
- Step 1: Pass to subsequences, use Cantor diagonalization $\to$ we can find $\mathcal{N} \subseteq \mathbb{N}$ infinte such that

$$\text{for each dyadic } \Gamma \subseteq [0,1), \quad (\langle r_n^\Gamma, D r_n^\Gamma \rangle)_{n\in\mathcal{N}} \text{ converges.}$$

- Step 2: Inductively construct a faithful Haar system $(\hat{h}_I)_{I\in\mathcal{D}}$ with "frequencies" in $\mathcal{N}$ and *with random signs*:

$$\hat{h}_I = \sum_{K\in\mathcal{B}_I} \theta_K h_K, \qquad (\theta_K)_{K\in\mathcal{B}_I} \in \{\pm 1\}^{\mathcal{B}_I}$$

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- $\implies$ The entries $d_I^{\text{stab}}$ of $D^{\text{stab}} = ADB$ satisfy

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and the variance is small $\to$ choose a “good” realization of $(\theta_K)$.

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- Perturbation argument $\implies$ $cI_Y - ADB$ is small.
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- An operator $T\colon Y \to Y$ has *large diagonal* (w.r.t. the Haar basis) if

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**Theorem (R. Lechner and T. S. '23)**

*Let $Y$ be a Haar system Hardy space with $\|\cdot\|_Y \not\sim \|\cdot\|_{L^\infty}$. Then the identity $I_Y$ factors through all operators $T \in \mathcal{B}(Y)$ with large diagonal, i.e., $(h_I)_{I\in\mathcal{D}}$ has the factorization property in $Y$.*

- Analogous results for $\ell^p$-sums of Haar system Hardy spaces.

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  - In $L^p$, $1 < p < \infty$: Invert directly. $D^{-1} h_I = d_I^{-1} h_I$, and $D^{-1}$ is bounded (by unconditionality).
  - In $L^1$: Semenov-Uksusov [4] $\implies$ bounded Haar multipliers have small "variation". See Lechner-Müller-Motakis-Schlumprecht [2].
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