Dimension dependence of factorization problems

Structures in Banach Spaces

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Overview

1 Dimension dependence of factorization problems

T. Speckhofer. *Dimension dependence of factorization problems:*Haar system Hardy spaces. Studia Mathematica, to appear.

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- Dyadic intervals: $\mathcal{D} = \{[0,1),[0,\frac{1}{2}),[\frac{1}{2},1),[0,\frac{1}{4}),[\frac{1}{4},\frac{1}{2}),\dots\}$
- ullet $I^+=$ left half, $I^-=$ right half of $I\in\mathcal{D}$
- Define $h_I = \mathbb{1}_{I^+} \mathbb{1}_{I^-}$, $I \in \mathcal{D}$.
- A Haar system space X is the completion of $H = \operatorname{span}\{\mathbbm{1}_{[0,1)}, h_I : I \in \mathcal{D}\}$ under a norm $\|\cdot\|_X$ such that:
 - If $x,y\in H$ and |x|,|y| have the same distribution, then $\|x\|_X=\|y\|_X$. • $\|\mathbb{1}_{[0,1)}\|_X=1$.
- Examples: L^p , $1 \le p < \infty$, all separable rearrangement-invariant function spaces on [0,1)

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• Haar system Hardy space = completion of $\operatorname{span}\{h_I: I \in \mathcal{D}\}$ under a rearrangement-invariant norm $\|\cdot\|_X$ or under $\|\cdot\|_{\circ}$, where

$$\left\| \sum_{I \in \mathcal{D}} a_I h_I \right\|_{\circ} = \left\| \left(\sum_{I \in \mathcal{D}} a_I^2 h_I^2 \right)^{1/2} \right\|_{X}.$$

- If $X = L^1$, then $\| \cdot \|_{\circ} = \| \cdot \|_{H^1}$
- From now on, let Y be a fixed Haar system Hardy space.

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- For $n \in \mathbb{N}_0$, let $Y_n = \operatorname{span}\{h_I : |I| \ge 2^{-n}\} \subset Y$.
- Given $n \in \mathbb{N}_0$, $\delta > 0$, how large does $N \in \mathbb{N}_0$ have to be chosen such that for every operator $T \colon Y_N \to Y_N$ with $\|T\| \le 1$ and with δ -large positive diagonal, there exists a factorization

$$\begin{array}{ccc} Y_n & \xrightarrow{I_{Y_n}} & Y_n \\ \downarrow & & \uparrow_A \\ Y_N & \xrightarrow{T} & Y_N \end{array}$$

where $||A|| ||B|| \le (1 + \varepsilon)/\delta$? (\rightarrow factorization constant)

ullet Variant: no "large diagonal", but factorization through T or $I_{Y_N}-T$.

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Dimension dependence: Introduction

- For $n \in \mathbb{N}_0$, let $Y_n = \operatorname{span}\{h_I : |I| > 2^{-n}\} \subset Y$.
- Given $n \in \mathbb{N}_0, \delta > 0$, how large does $N \in \mathbb{N}_0$ have to be chosen such that for every operator $T: Y_N \to Y_N$ with ||T|| < 1 and with δ -large positive diagonal, there exists a factorization

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- In L^p , $1 \le p \le \infty$: Restricted Invertibility Theorem (Bourgain-Tzafriri 1987)
 - → linear dimension dependence
- ullet Conversely: factorization $\implies T$ "well invertible" on a large subspace
- Bourgain's localization method may yield primariness
- Results in other classical spaces: $\mathcal{B}(\ell^2)$ (Blower), H^1 and BMO (Müller), $\ell^\infty(L^p)$ (Wark), . . .
- ullet Existing bounds for N are often super-exponential functions of n
- Lechner 2019: $N \geq Cn$ is sufficient in H^p , $1 \leq p < \infty$, and SL^{∞} .

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Theorem (S. '24)

Let Y be a Haar system Hardy space and $\varepsilon > 0$. Put $\eta = \frac{\varepsilon}{6(1+\varepsilon)}$. If

$$N \ge 42n(n+1)\left\lceil \frac{1}{\eta} \right\rceil + 42 + \left\lfloor 4\log_2\left(\frac{1}{\eta}\right) \right\rfloor,\tag{1}$$

then for every linear operator $T: Y_N \to Y_N$ with $||T|| \le 1$, the identity I_{Y_n} factors through T or $I_{Y_N} - T$ with constant $2(1 + \varepsilon)$.

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If $(h_I)_I$ is K-unconditional in Y, then (1) can be replaced by

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Corollary (S. '24)

If $N \geq C n^4 2^{C n^2}$, where $C = C(\eta, \delta)$, then the word "positive" can be omitted (this doubles the factorization constant).

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- Basic idea: Step-by-step reduction
 - Operator $T \to \mathsf{Haar}$ multiplier $D \to \mathsf{constant}$ multiple of the identity cI_{Y_n}
- ullet Clearly, the identity I_{Y_n} factors through cI_{Y_n} or $(1-c)I_{Y_n}$

$$D \approx A_1 T B_1, \qquad c I_{Y_n} \approx A_2 D B_2$$

• How are A_i, B_i defined? \rightarrow faithful Haar system $(\hat{h}_I)_I$ Associated operators A, B:

$$Bx = \sum_{I} \frac{\langle h_{I}, x \rangle}{|I|} \hat{h}_{I}, \qquad Ax = \sum_{I} \frac{\langle \hat{h}_{I}, x \rangle}{|I|} h_{I}$$

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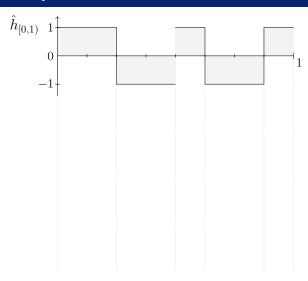
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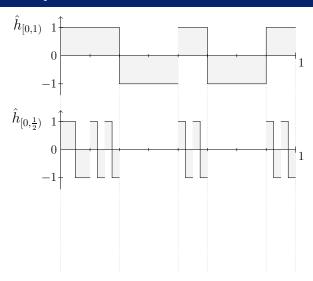
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Faithful Haar system



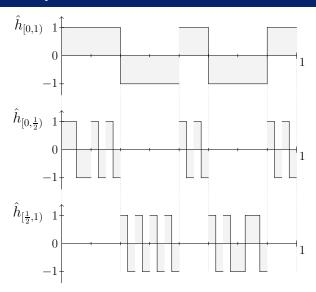
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Faithful Haar system



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First step: diagonalization via random faithful Haar systems

- Choose signs $\theta = (\theta_K)_{K \in \mathcal{D}} \in \{\pm 1\}^{\mathcal{D}}$ uniformly at random.
- Construct a (finite) randomized faithful Haar system by

$$\hat{h}_I(\theta) = \sum_{K \in \mathcal{B}_I(\theta)} \theta_K h_K, \qquad |I| \ge 2^{-Cn^2},$$

where $\mathcal{B}_I(\theta) \subset \mathcal{D}$

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First step: diagonalization via random faithful Haar systems

- Choose signs $\theta = (\theta_K)_{K \in \mathcal{D}} \in \{\pm 1\}^{\mathcal{D}}$ uniformly at random.
- Construct a (finite) randomized faithful Haar system by

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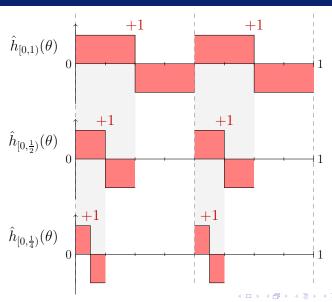
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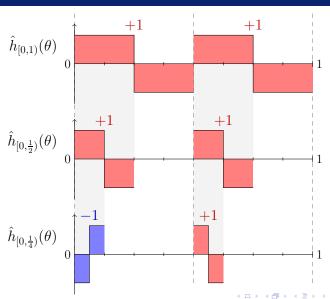
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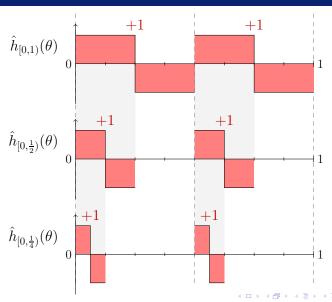
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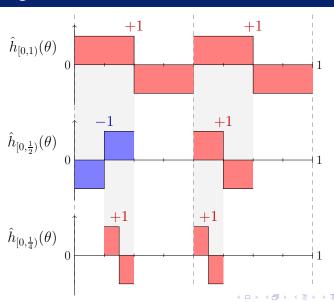
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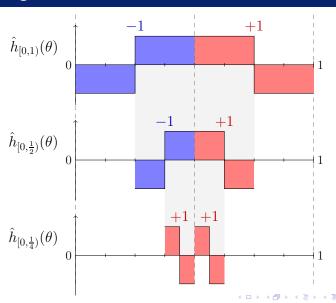




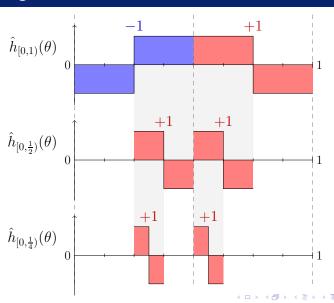




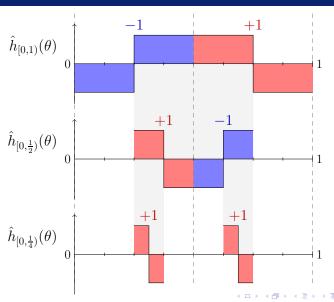
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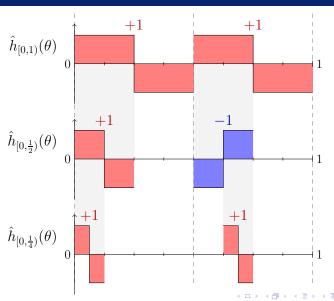


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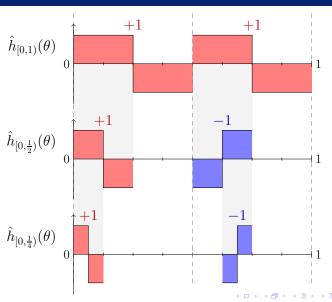


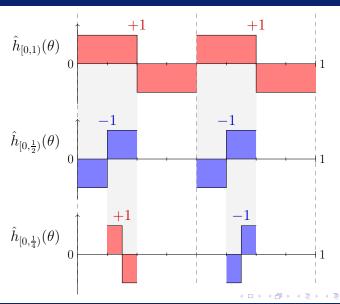
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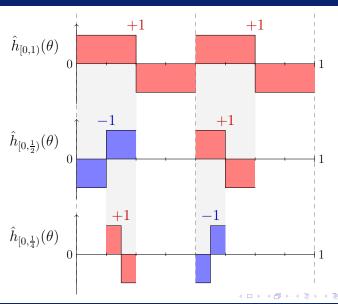


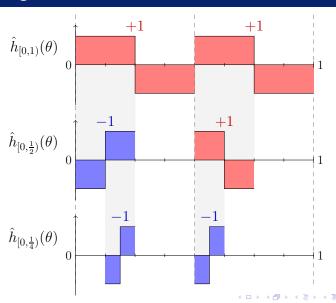


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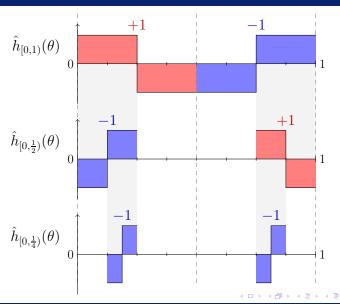


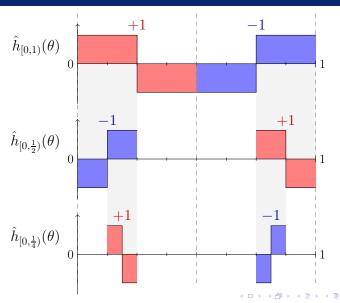


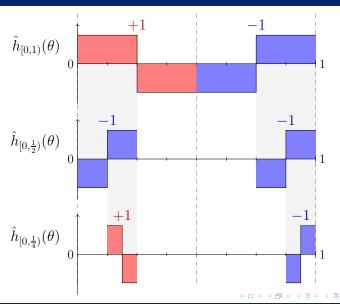


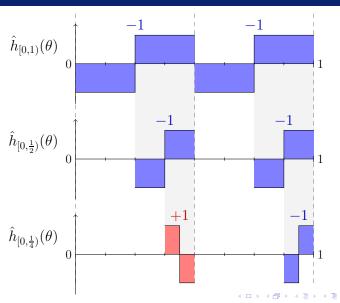


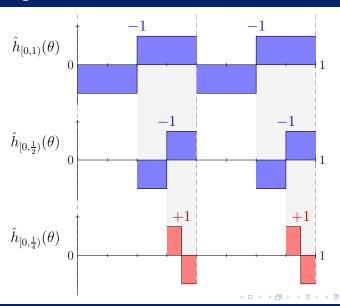
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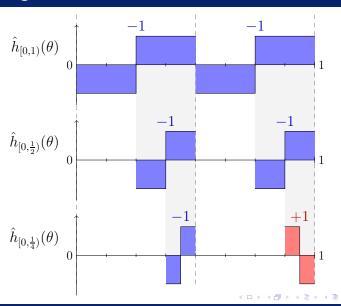




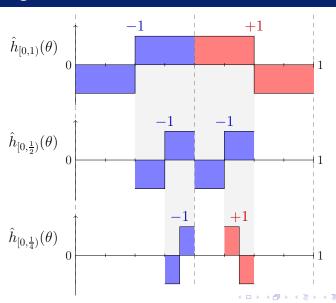


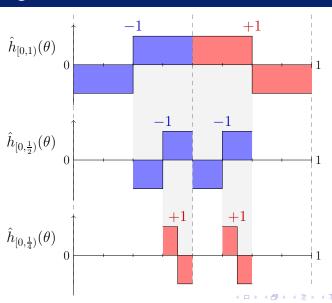


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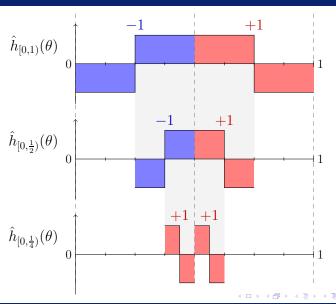


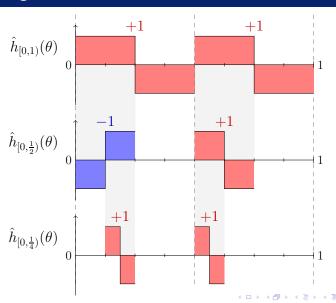
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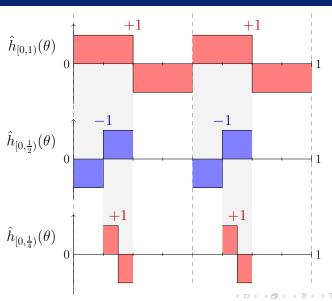


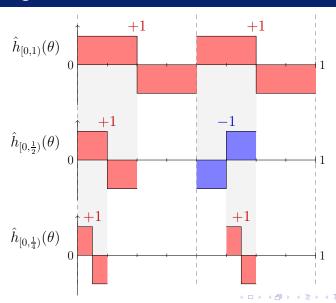


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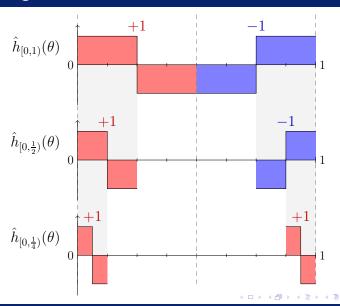




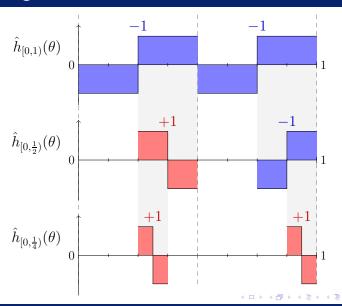


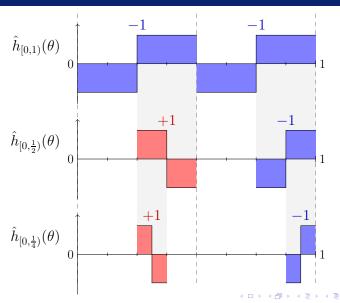


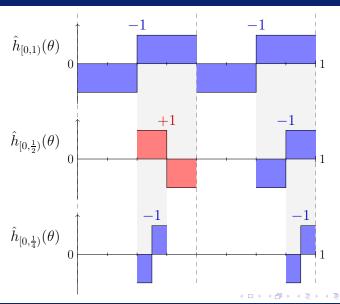
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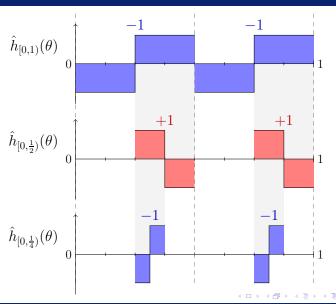


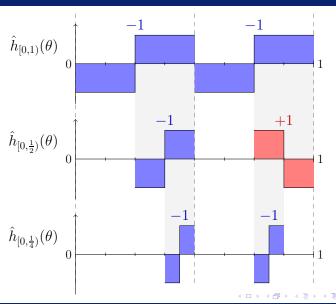
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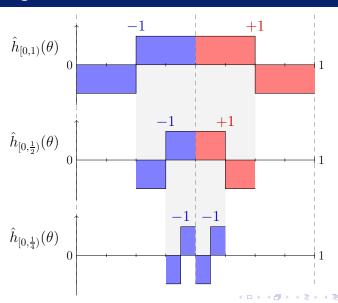


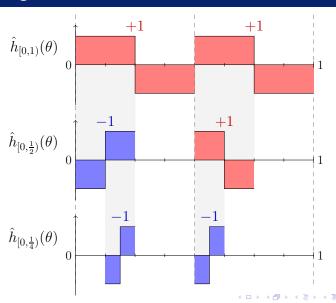


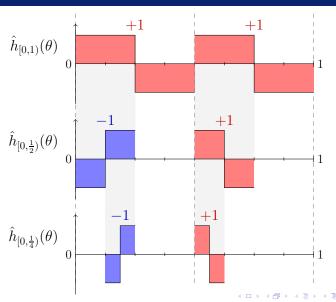


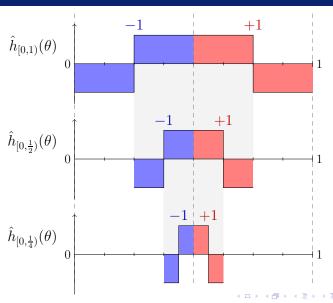


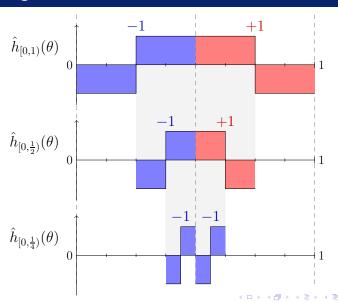


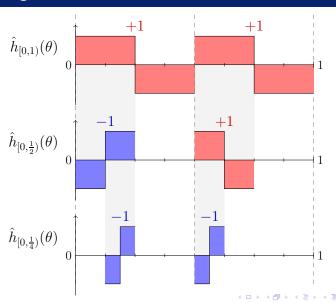


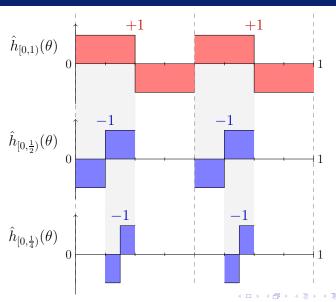


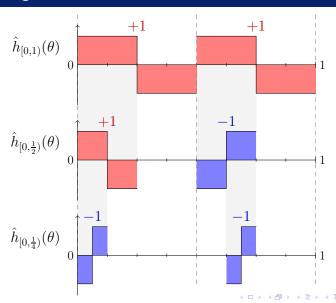


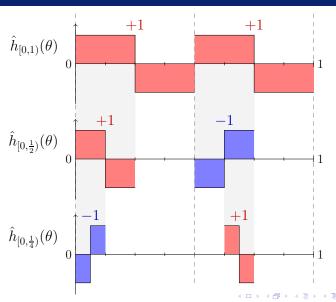












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$$\mathbb{E}X_{I,J} = 0$$
 for $I \neq J$ and $\mathbb{V}X_{I,J} \leq 3||T||^2 2^{-m/2}$

(m = first level used in our construction)

• Choose m large \implies for some realization of θ , the system almost diagonalizes T.

Second step: stabilization of Haar multipliers

- ullet Above, we obtain a Haar multiplier D with entries $(d_I)_I$
- ullet D is stable along every level: $d_I pprox d_J$ whenever |I| = |J|
- ullet Use pigeonhole principle to stabilize across all levels $\leadsto cI_{Y_n}$

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Thank you for your attention!



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