Dimension dependence of factorization problems

Structures in Banach Spaces

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Overview

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Definitions

- Dyadic intervals: $\mathcal{D} = \{[0,1), [0,\frac{1}{2}), [\frac{1}{2},1), [0,\frac{1}{4}), [\frac{1}{4},\frac{1}{2}), \dots\}$
- $I^+ =$ left half, $I^- =$ right half of $I \in \mathcal{D}$
- Define $h_I = \mathbb{1}_{I^+} - \mathbb{1}_{I^-}$, $I \in \mathcal{D}$.
- A **Haar system space** X is the completion of $H = \operatorname{span}\{\mathbb{1}_{[0,1)}, h_I : I \in \mathcal{D}\}$ under a norm $\|\cdot\|_X$ such that:
  - If $x, y \in H$ and $|x|$, $|y|$ have the same distribution, then $\|x\|_X = \|y\|_X$.
  - $\|\mathbb{1}_{[0,1)}\|_X = 1$.
- Examples: L^p, $1 \le p < \infty$, all separable rearrangement-invariant function spaces on $[0,1)$

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$$\Big\|\sum_{I\in\mathcal{D}} a_I h_I\Big\|_\circ = \Big\|\Big(\sum_{I\in\mathcal{D}} a_I^2 h_I^2\Big)^{1/2}\Big\|_X.$$

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- For $n \in \mathbb{N}_0$, let $Y_n = \operatorname{span}\{h_I : |I| \geq 2^{-n}\} \subset Y$.
- Given $n \in \mathbb{N}_0, \delta > 0$, how large does $N \in \mathbb{N}_0$ have to be chosen such that for every operator $T\colon Y_N \to Y_N$ with $\|T\| \leq 1$ and with δ-large positive diagonal, there exists a factorization

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- In L^p, $1 \leq p \leq \infty$:
  Restricted Invertibility Theorem (Bourgain-Tzafriri 1987)
  $\rightarrow$ linear dimension dependence
- Conversely: factorization $\implies$ T "well invertible" on a large subspace
- Bourgain's localization method may yield primariness
- Results in other classical spaces:
  $\mathcal{B}(\ell^2)$ (Blower), H^1 and BMO (Müller), $\ell^\infty(L^p)$ (Wark), ...
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Main results

Theorem (S. '24)

Let Y be a Haar system Hardy space and $\varepsilon > 0$. Put $\eta = \frac{\varepsilon}{6(1+\varepsilon)}$. If

$$N \geq 42n(n+1)\left\lceil \frac{1}{\eta} \right\rceil + 42 + \left\lfloor 4 \log_2\left(\frac{1}{\eta}\right) \right\rfloor, \tag{1}$$

then for every linear operator $T \colon Y_N \to Y_N$ with $\|T\| \leq 1$, the identity I_{Y_n} factors through T or $I_{Y_N} - T$ with constant $2(1+\varepsilon)$.

If $(h_I)_I$ is K-unconditional in Y, then (1) can be replaced by

$$N \geq C(\eta, K)\, n.$$

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Let Y be a Haar system Hardy space and $\delta, \varepsilon > 0$. Put $\eta = \frac{\varepsilon\delta}{6(1+\varepsilon)}$. If

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Corollary (S. '24)

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Proof method

- Basic idea: Step-by-step reduction
  Operator $T \to$ Haar multiplier $D \to$ constant multiple of the identity cI_{Y_n}
- Clearly, the identity I_{Y_n} factors through cI_{Y_n} or $(1-c)I_{Y_n}$.

$$D \approx A_1 T B_1, \qquad cI_{Y_n} \approx A_2 D B_2$$

- How are A_i, B_i defined? $\to$ *faithful Haar system* $(\hat{h}_I)_I$, Associated operators A, B:

$$Bx = \sum_I \frac{\langle h_I, x\rangle}{|I|} \hat{h}_I, \qquad Ax = \sum_I \frac{\langle \hat{h}_I, x\rangle}{|I|} h_I$$

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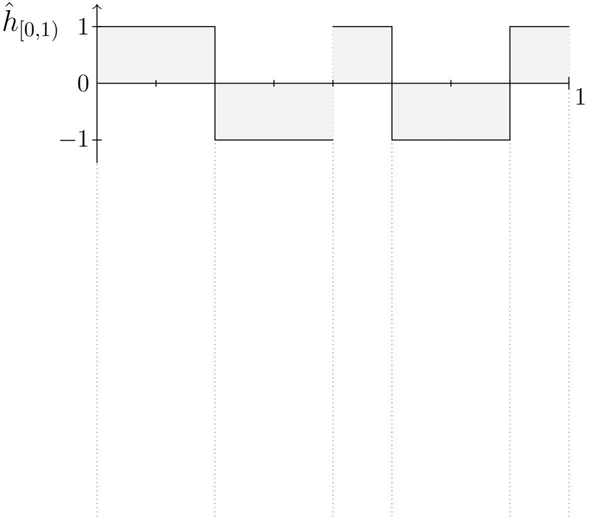
Faithful Haar system



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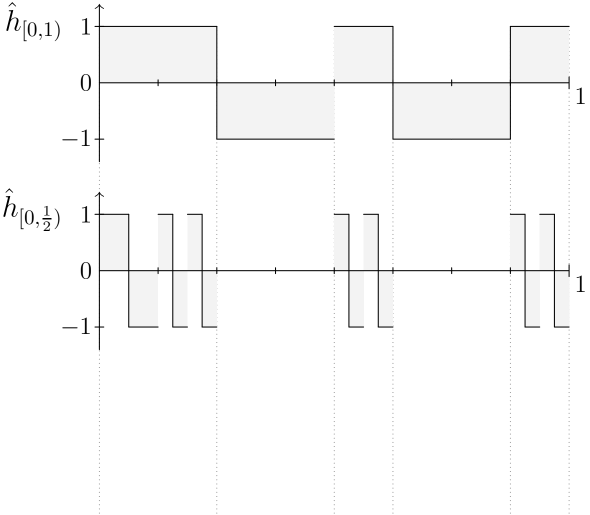

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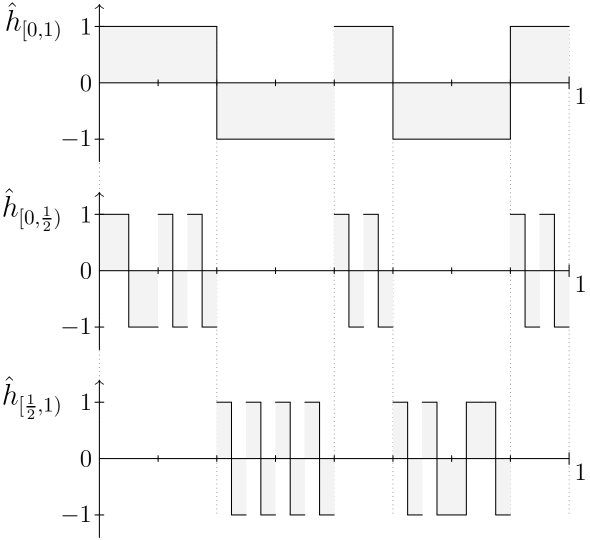

Proof: Diagonalization

First step: ***diagonalization via random faithful Haar systems***

- Choose signs $\theta = (\theta_K)_{K\in\mathcal{D}} \in \{\pm 1\}^{\mathcal{D}}$ uniformly at random.
- Construct a (finite) randomized faithful Haar system by

$$\hat{h}_I(\theta) = \sum_{K\in\mathcal{B}_I(\theta)} \theta_K h_K, \qquad |I| \geq 2^{-Cn^2},$$

where $\mathcal{B}_I(\theta) \subset \mathcal{D}$.

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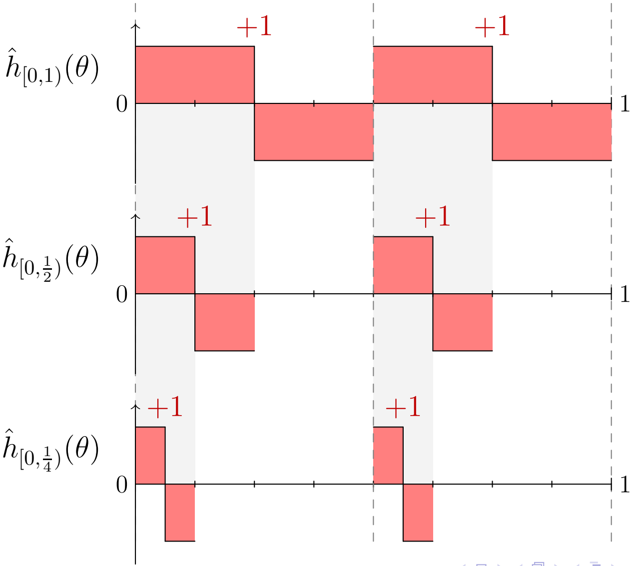

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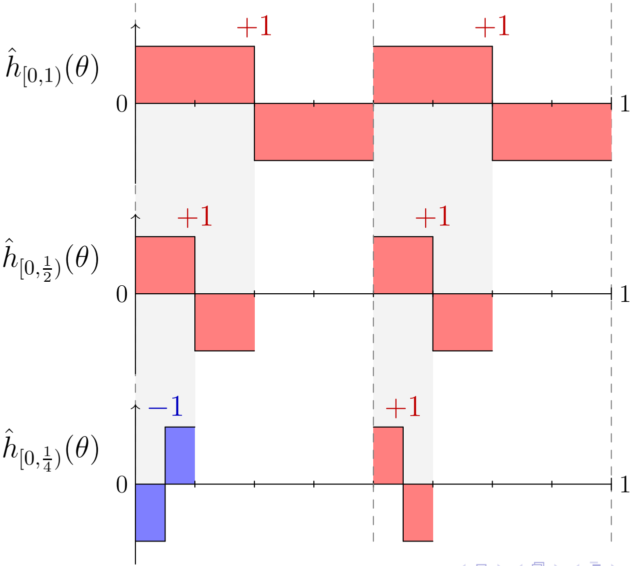

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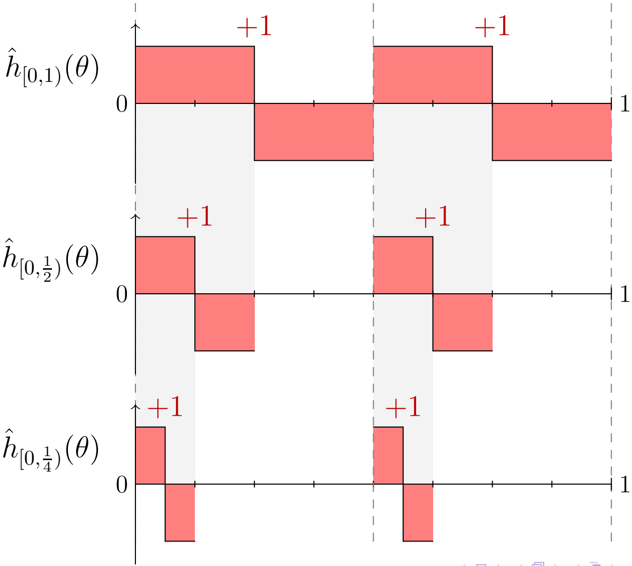

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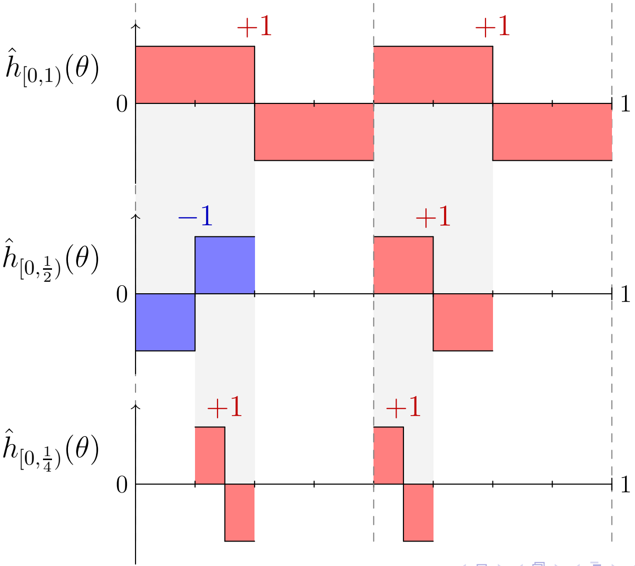

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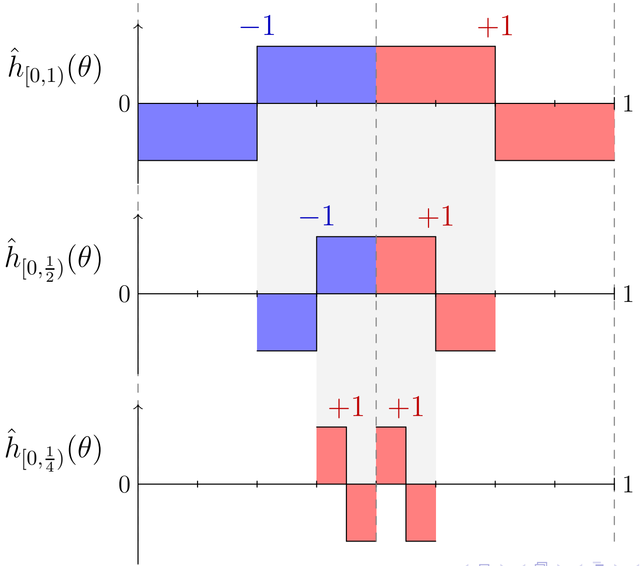

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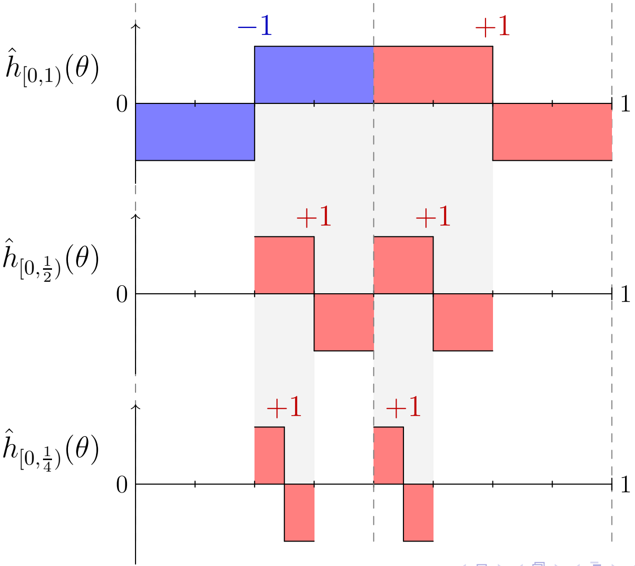

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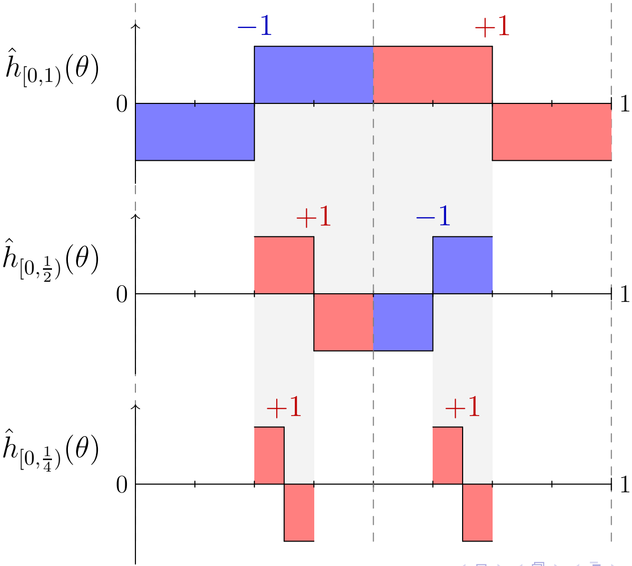

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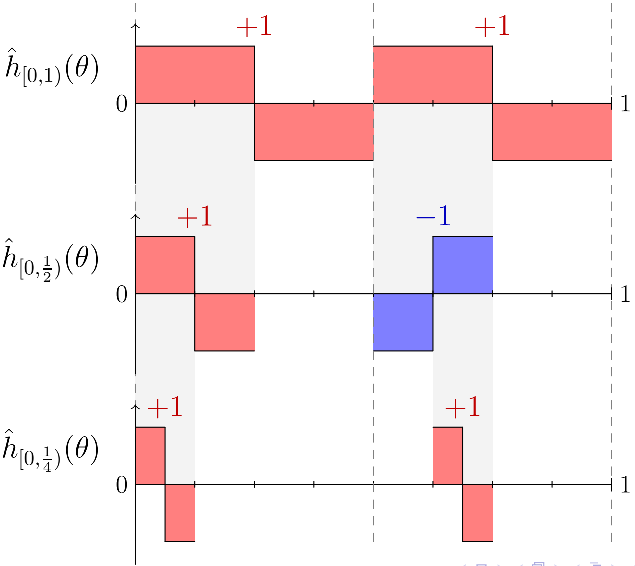

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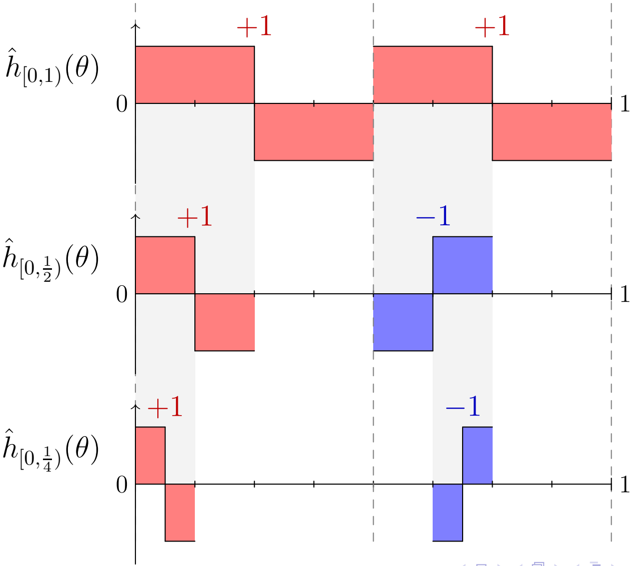

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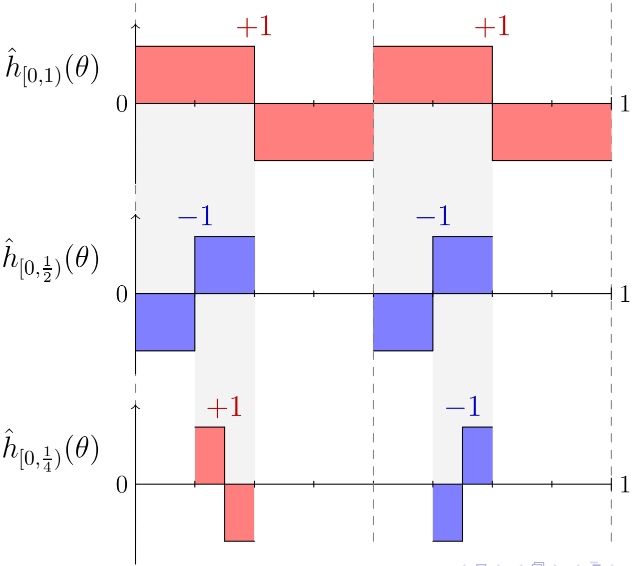

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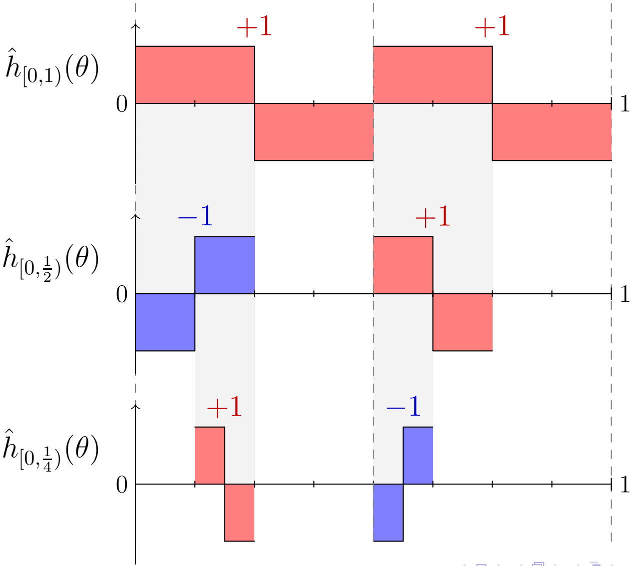

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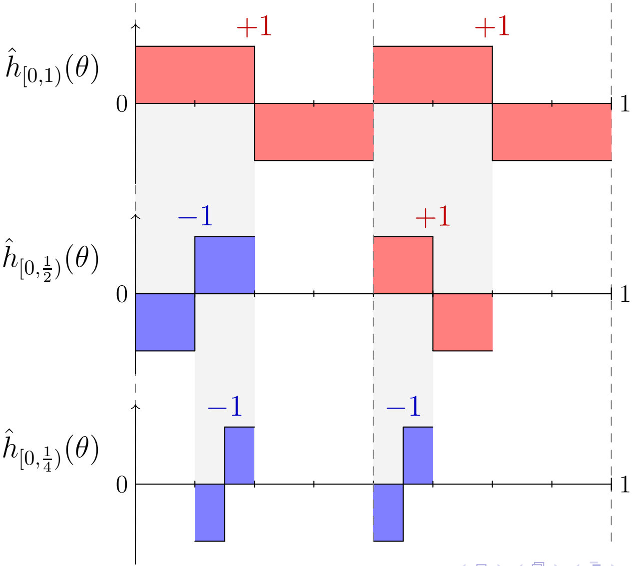

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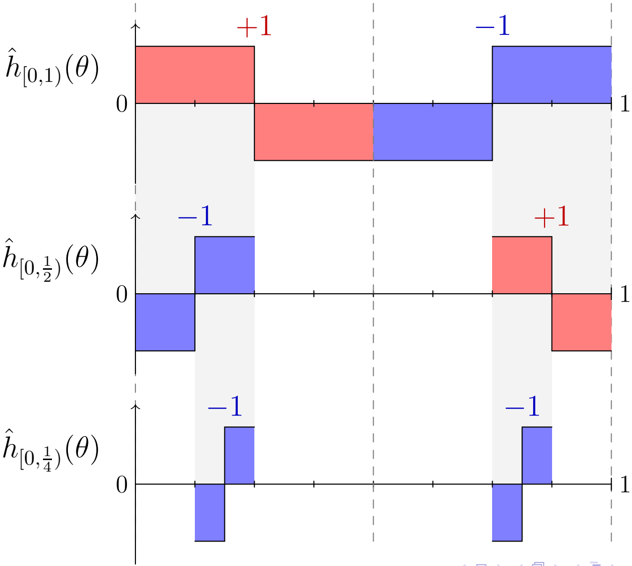

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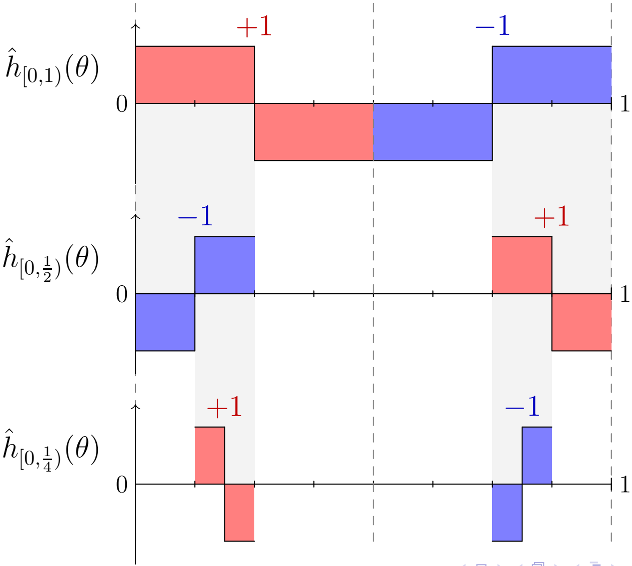

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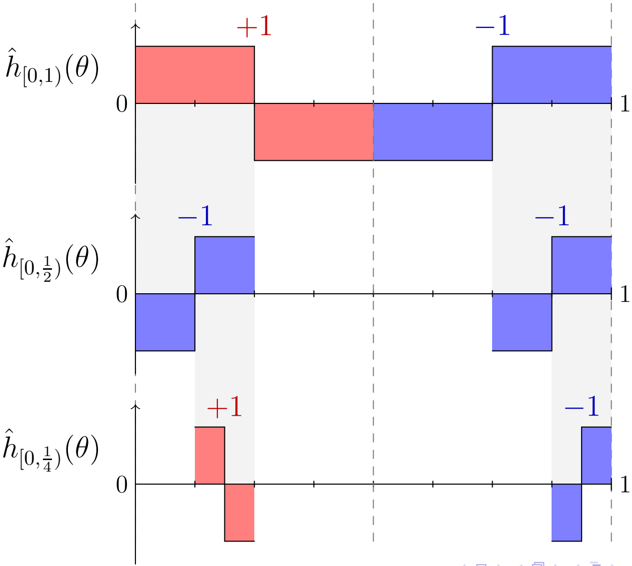

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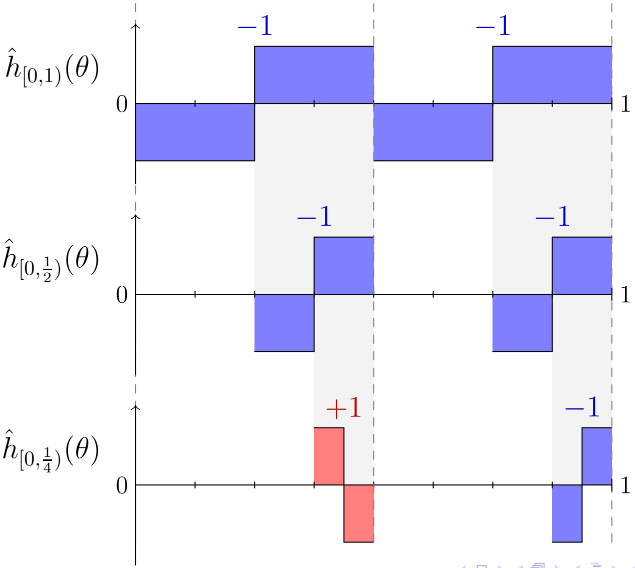

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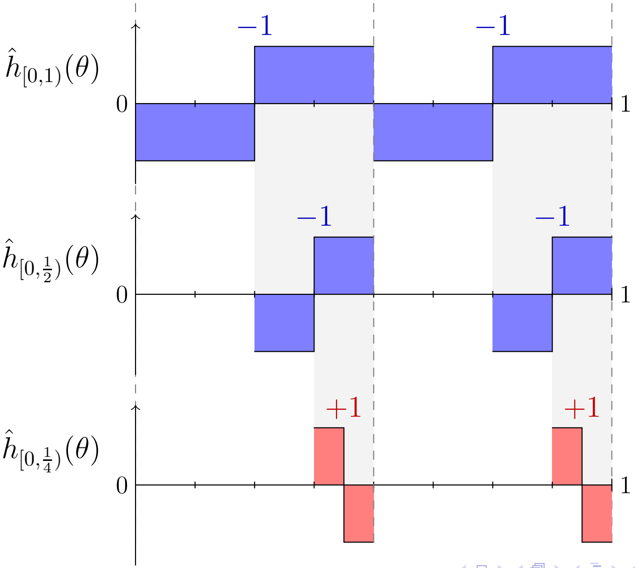

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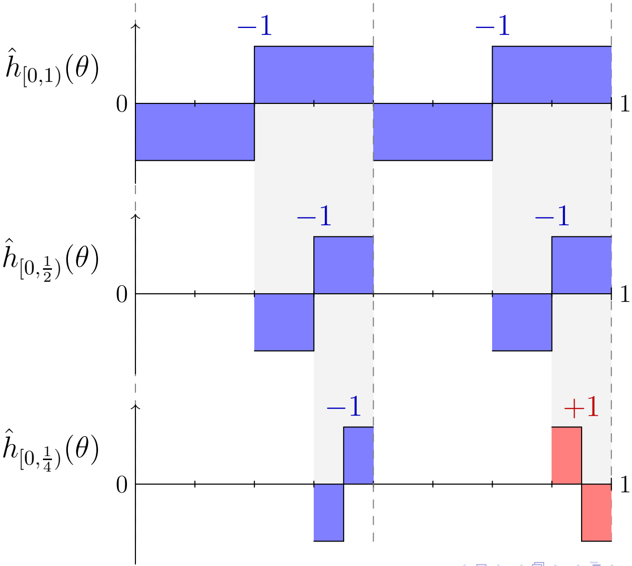

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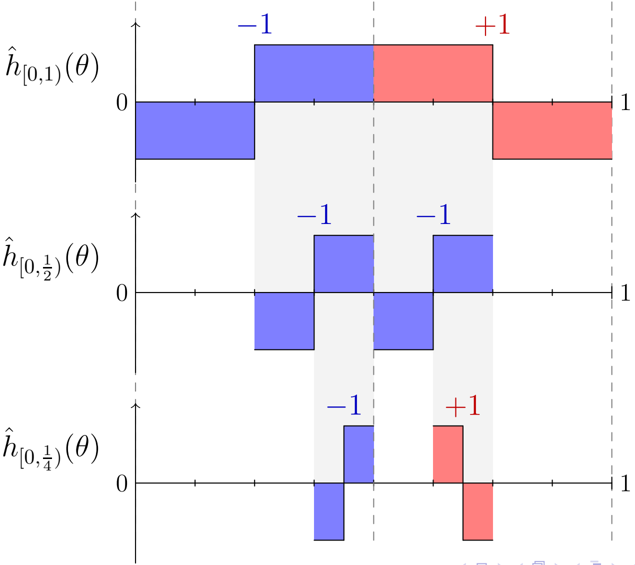

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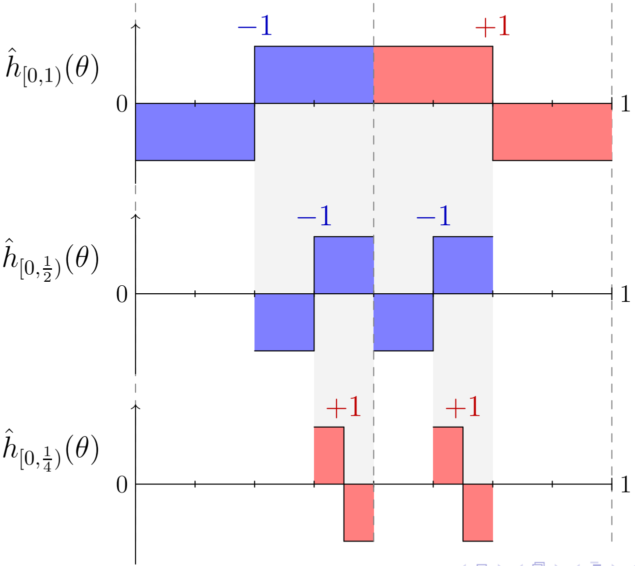

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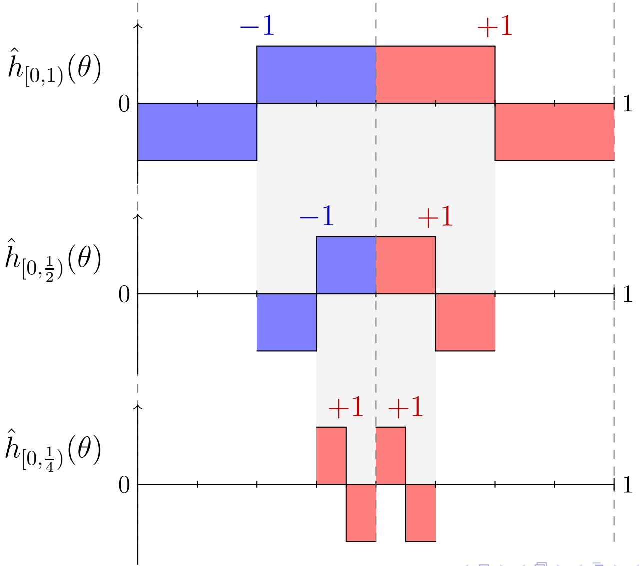

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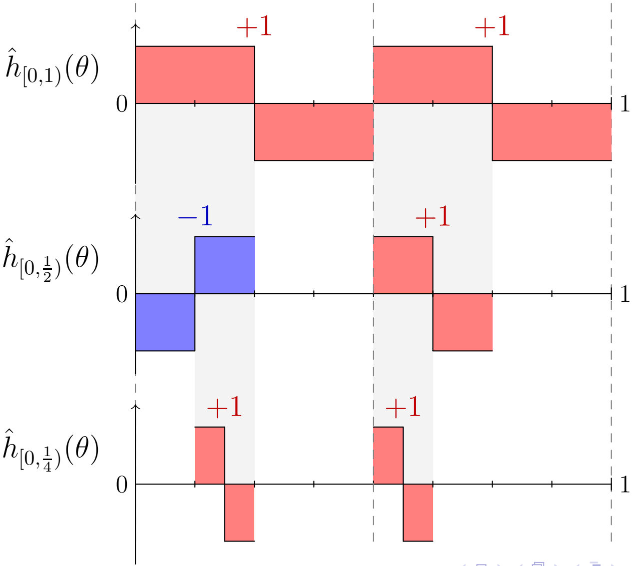

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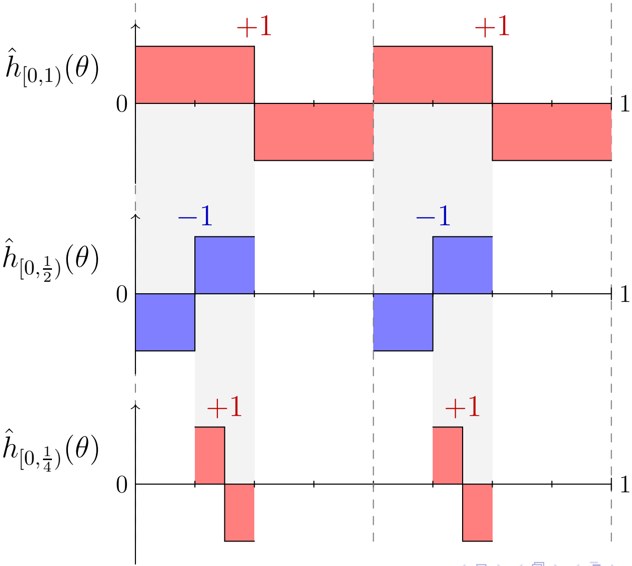

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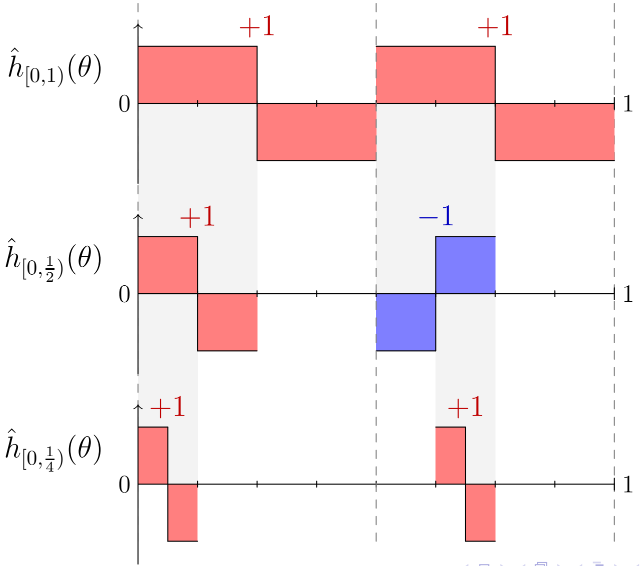

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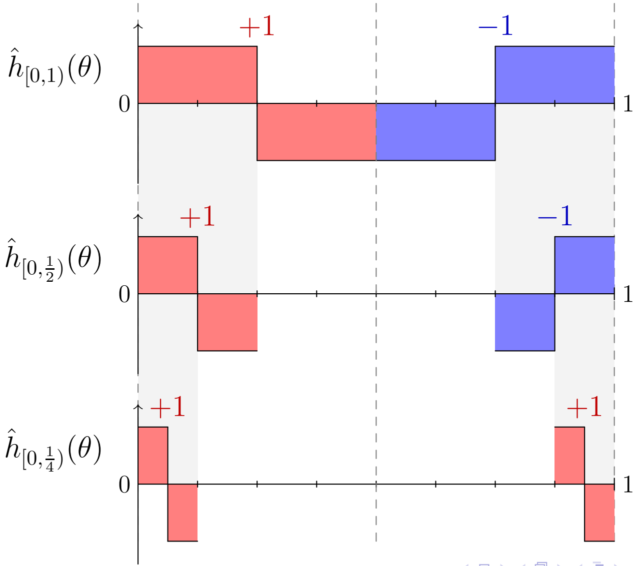

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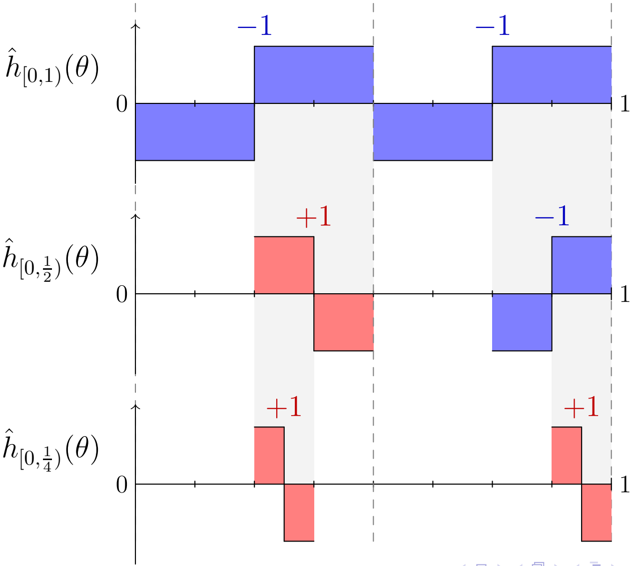

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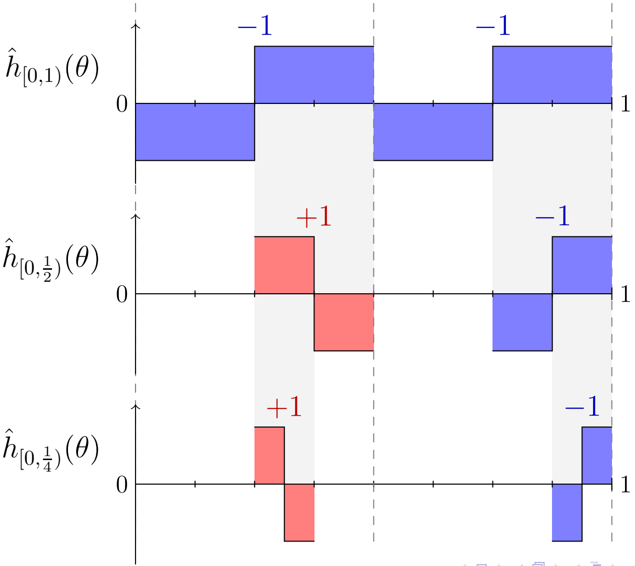

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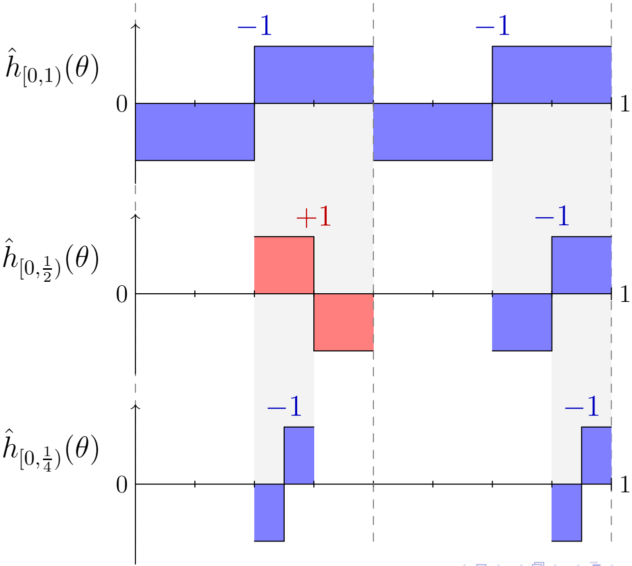

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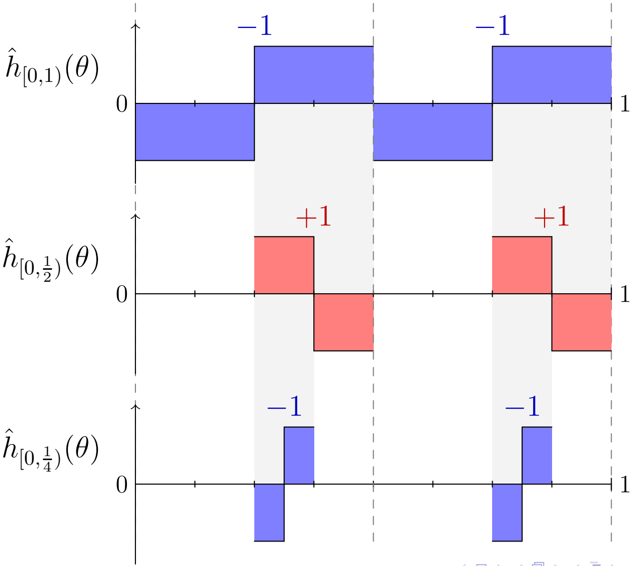

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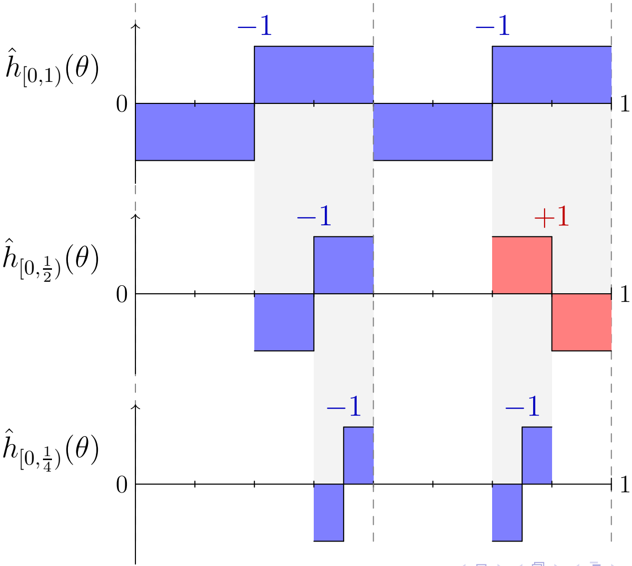

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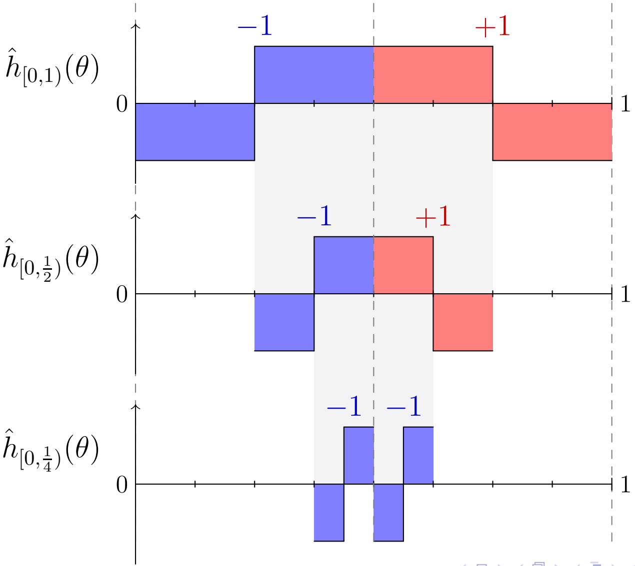

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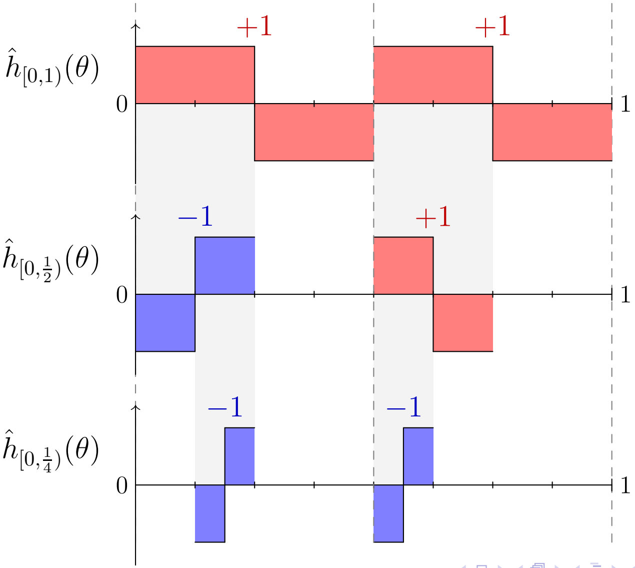

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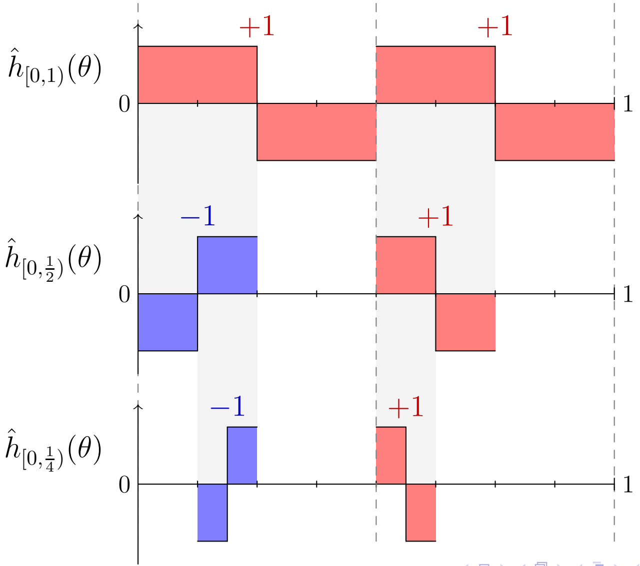

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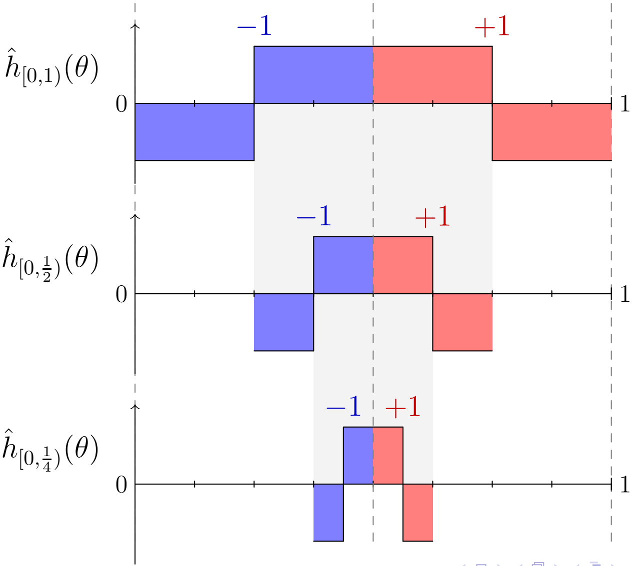

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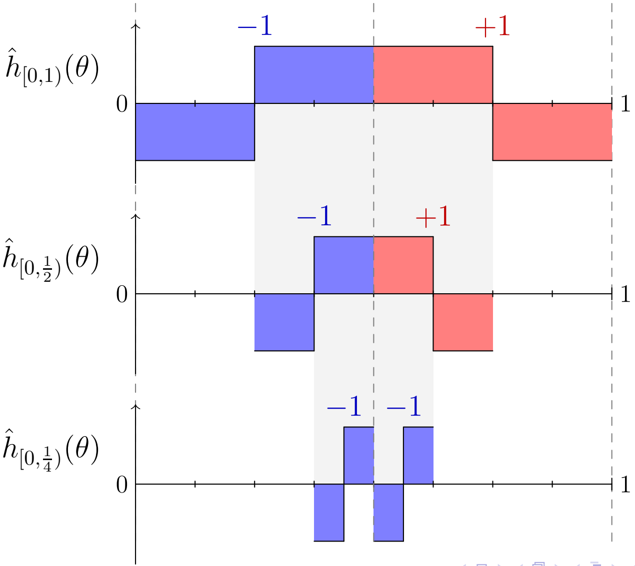

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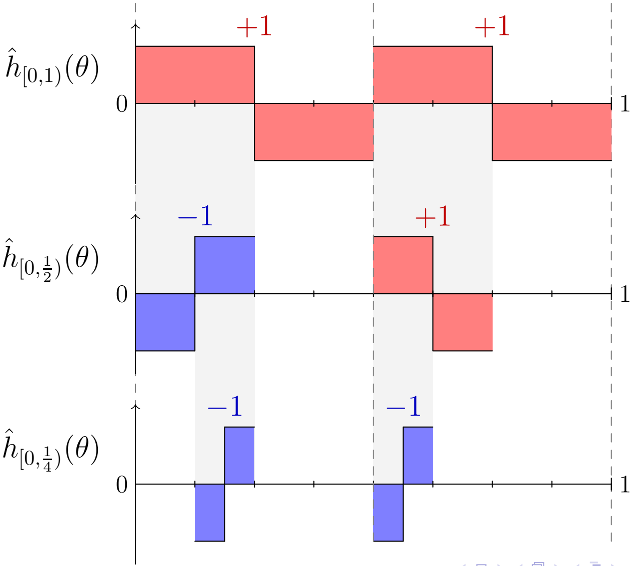

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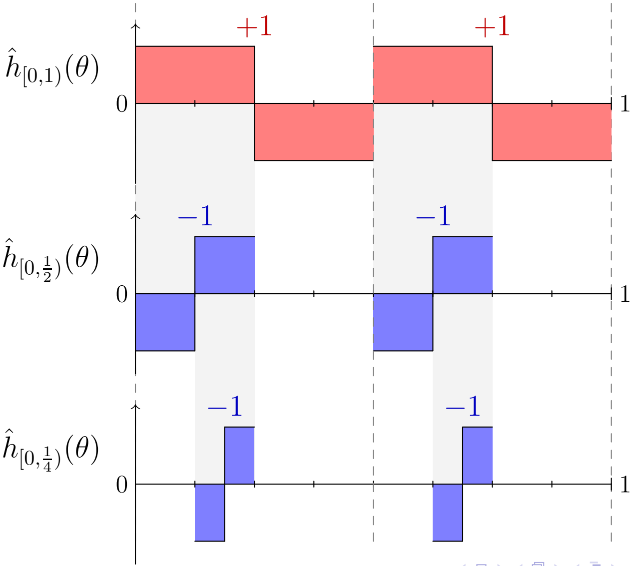

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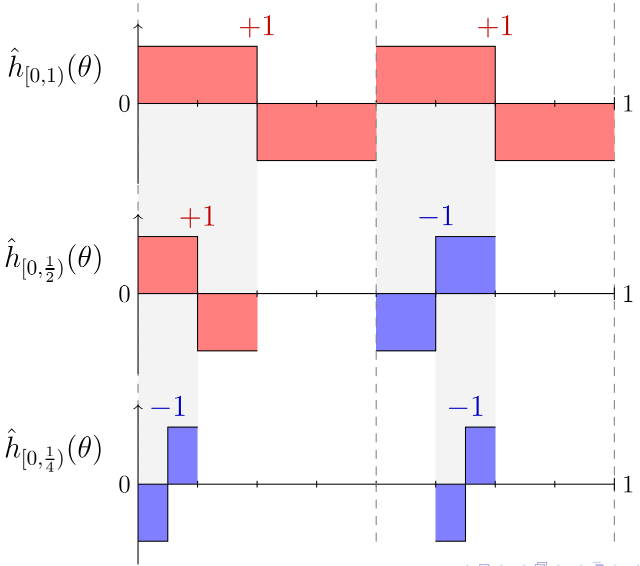

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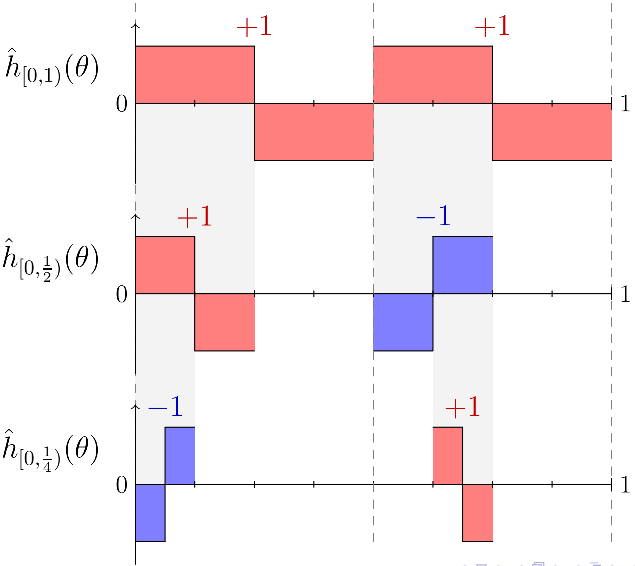

Proof: Diagonalization and stabilization

- The random variables $X_{I,J}(\theta) = \langle \hat{h}_I(\theta), T\hat{h}_J(\theta) \rangle$ satisfy

$$\mathbb{E}X_{I,J} = 0 \text{ for } I \neq J \qquad \text{and} \qquad \mathbb{V}X_{I,J} \leq 3\|T\|^2 2^{-m/2}$$

($m =$ first level used in our construction)

- Choose m large $\implies$ for some realization of θ, the system almost diagonalizes T.

Second step: ***stabilization of Haar multipliers***

- Above, we obtain a Haar multiplier D with entries $(d_I)_I$.
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