Problem 12163. *Proposed by Thomas Speckhofer, Attnang-Puchheim, Austria.*
Let $\mathbb{R}^n$ have the usual dot product and norm. When $v = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let
$\Sigma v = x_1 + \cdots + x_n$. Prove

$$\|v\|^2 \|w\|^2 \geq (v \cdot w)^2 + \frac{1}{n}(\|v\| |\Sigma w| - \|w\| |\Sigma v|)^2$$

for all $v, w \in \mathbb{R}^n$.