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Dimension dependence of factorization problems: Haar system Hardy spaces

by

Thomas Speckhofer

Abstract. For $n \in \mathbb{N}$, let Y_n denote the linear span of the first $n+1$ levels of the Haar system in a *Haar system Hardy space* Y (this class contains all separable rearrangement-invariant function spaces and also related spaces such as dyadic H^1). Let I_{Y_n} denote the identity operator on Y_n. We prove the following quantitative factorization result: Fix $\Gamma, \delta, \varepsilon > 0$, and let $n, N \in \mathbb{N}$ be chosen such that $N \geq Cn^2$, where $C = C(\Gamma, \delta, \varepsilon) > 0$ (this amounts to a quasi-polynomial dependence between $\dim Y_N$ and $\dim Y_n$). Then for every linear operator $T\colon Y_N \to Y_N$ with $\|T\| \leq \Gamma$, there exist operators A, B with $\|A\|\,\|B\| \leq 2(1+\varepsilon)$ such that either $I_{Y_n} = ATB$ or $I_{Y_n} = A(I_{Y_N} - T)B$. Moreover, if T has δ-large positive diagonal with respect to the Haar system, then $I_{Y_n} = ATB$ for some A, B with $\|A\|\,\|B\| \leq (1+\varepsilon)/\delta$. If the Haar system is unconditional in Y, then an inequality of the form $N \geq Cn$ is sufficient for the above statements to hold (hence, $\dim Y_N$ depends polynomially on $\dim Y_n$). Finally, we prove an analogous result in the case where T has large but not necessarily positive diagonal entries.

1. Introduction and main results. This article deals with quantitative factorization problems in finite-dimensional Banach spaces. We consider problems of the following form:

Problem 1.1. *Let $X_0 \subset X_1 \subset \cdots$ be a sequence of finite-dimensional subspaces of a Banach space X, let $(e_j)_{j=1}^{\infty}$ be a sequence in X, and assume that for every $n \geq 0$, $(e_j)_{j=1}^{d_n}$ is a basis of X_n (where $d_n = \dim X_n$). Given $\Gamma, \delta, \varepsilon > 0$, find conditions on $n, N \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that the following statement holds: For every linear operator $T\colon X_N \to X_N$ with $\|T\| \leq \Gamma$ and with δ-large positive diagonal with respect to $(e_j)_{j=1}^{d_N}$ (i.e., all diagonal entries of the matrix of T with respect to $(e_j)_{j=1}^{d_N}$ are greater than or equal to δ), there exist linear operators $B\colon X_n \to X_N$ and $A\colon X_N \to X_n$ with*

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$\|A\|\,\|B\| \le \frac{1+\varepsilon}{\delta}$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} X_n & \xrightarrow{I_{X_n}} & X_n \\ {\scriptstyle B}\downarrow & & \uparrow{\scriptstyle A} \\ X_N & \xrightarrow{T} & X_N \end{array}$$

In that case, we say that the identity I_{X_n} factors through T with constant $\frac{1+\varepsilon}{\delta}$.

We will also consider variations of Problem 1.1 where the condition on the diagonal of T is removed and the conclusion is replaced by a factorization through T or $I_{X_N} - T$, or where the diagonal entries only have to be large *in absolute value.*

In the following, we will always take X_n to be the subspace spanned by the first $n+1$ levels of the Haar system in a fixed underlying space Y. Thus, we put $X_n = Y_n := \operatorname{span}\{h_I : |I| \ge 2^{-n}\}$, where $(h_I)_{I\in\mathcal{D}}$ denotes the *Haar system* on $[0,1)$, indexed by the dyadic intervals $I \in \mathcal{D}$ (and we equip every space Y_n with its Haar basis). In our results, the space Y will be a *Haar system Hardy space.* This class of Banach spaces was introduced in [18, 16], and the precise definition will be given in Section 2.2. For now, we only mention that Y is the completion of the linear span of the Haar system on $[0,1)$, either under a rearrangement-invariant norm (such as the L^p-norm, where $1 \le p \le \infty$, or an Orlicz norm) or under an associated norm that can be defined via the square function—hence, the dyadic Hardy space H^1, equipped with an equivalent norm, is also contained in this class. Factorization results for these *infinite-dimensional* spaces have been proved by Kh. V. Navoyan [24] and by R. Lechner and the author [18]. Moreover, in [16], R. Lechner, P. Motakis, P. F. X. Müller and Th. Schlumprecht proved factorization results for Haar multipliers on *bi-parameter* Haar system Hardy spaces.

Note that in the special case where $X_n = Y_n \subset L^p$ for some $1 \le p \le \infty$, Problem 1.1 can be solved by applying the *Restricted Invertibility Theorem* of J. Bourgain and L. Tzafriri [4] (see also [27, 23, 19]): If R_N is the coordinate projection from Y_N onto $Z_N := \operatorname{span}\{h_I : |I| = 2^{-N}\}$, then $R_N T|_{Z_N}$ can be identified with an operator on $\ell^p_{2^N}$ with large positive diagonal (with respect to the unit vector basis). By [4, Corollary 3.2], this operator is *well invertible* on a large subspace of $\ell^p_{2^N}$ spanned by $2^n \ge c2^N$ basis elements, where $c > 0$ does not depend on N (see also [22, Section 5] for more details). Since Y_n can be identified with (a subspace of) $\ell^p_{2^n}$, this yields a factorization of I_{Y_n} through T. Hence, in this case we have linear dimension dependence. Conversely, note that if $I_{Y_n} = ATB$ for some operators A, B with $\|A\|\,\|B\| \le C$, then T is well invertible on the image of B. Related results on factorization and restricted invertibility of *continuous matrix functions* can be found in [5, 6].

Quantitative bounds for N (in terms of n) in Problem 1.1 have also been proved for many other classical Banach spaces, and by employing *Bourgain's localization method* [3], it is sometimes possible to prove the primarity of a Banach space by reducing this problem to a finite-dimensional factorization problem. We refer to [20, 2, 1, 28, 29, 22, 17, 10, 11, 13] for such factorization and primarity results. In spaces with less structure than L^p, however, the best known lower bounds on N are often super-exponential functions of n.

In the case where Y is the dyadic Hardy space H^p, $1 \le p < \infty$, or SL^∞, R. Lechner [12] obtained a factorization result where N depends linearly on n, improving on previous results where the estimate for N was a nested exponential function of n. Our main result states that in every Haar system Hardy space, an inequality of the form $N \ge C(\Gamma, \delta, \varepsilon)n^2$ (where $C(\Gamma, \delta, \varepsilon) > 0$) is sufficient for the conclusion of Problem 1.1 and its variation involving T and $I_{Y_N} - T$ to hold. Moreover, if the Haar system is K-unconditional in the space under consideration for some $K \ge 1$, then $N \ge C(\Gamma, \delta, \varepsilon, K)n$ suffices. In the following, the variable δ will either denote a lower bound for the diagonal of an operator T, or it will be omitted (i.e., set to 1) when stating the result about factorizations through T or $I_{Y_N} - T$.

THEOREM 1.2. *Let Y be a Haar system Hardy space, and let $\Gamma, \delta, \varepsilon > 0$. Put $\eta = \frac{\varepsilon}{6(1+\varepsilon)}$. Moreover, let $n, N \in \mathbb{N}_0$ be chosen so that*

$$N \ge 42n(n+1)\left\lceil \frac{\Gamma}{\eta\delta} \right\rceil + 42 + \left\lfloor 4\log_2\left(\frac{\Gamma}{\eta\delta}\right)\right\rfloor. \tag{1.1}$$

Then for every linear operator $T\colon Y_N \to Y_N$ with $\|T\| \le \Gamma$, the following hold:

- (i) *If $\delta = 1$ in (1.1), then the identity I_{Y_n} factors through T or $I_{Y_N} - T$ with constant $2(1+\varepsilon)$.*
- (ii) *If T has δ-large positive diagonal with respect to the Haar basis, then the identity I_{Y_n} factors through T with constant $\frac{1+\varepsilon}{\delta}$.*

If the Haar system is K-unconditional in Y for some $K \ge 1$, then inequality (1.1) can be replaced by

$$N \ge 42n\left\lceil \frac{K\Gamma}{\eta\delta} \right\rceil + 42 + \left\lfloor 4\log_2\left(\frac{\Gamma}{\eta\delta}\right)\right\rfloor. \tag{1.2}$$

In the following, we will denote the right-hand side of (1.1) by $N_{\min}(\Gamma, \delta, \varepsilon, n)$ and the right-hand side of (1.2) by $N_{\min}^K(\Gamma, \delta, \varepsilon, n)$.

REMARK 1.3. Due to (1.1), we have a quasi-polynomial dependence between $\dim Y_N \sim 2^N$ and $\dim Y_n \sim 2^n$ in the general case, whereas by (1.2), we have polynomial dimension dependence in the unconditional case.

We will prove Theorem 1.2 in Section 4. The basic structure of our proof is similar to that of the proofs in [12, 18]: Starting with a given operator T, we perform a step-by-step reduction which yields "simpler" operators. First, we use a modified version of the probabilistic argument in [12] to *diagonalize* the operator T (up to a small error). Next, we *stabilize* the resulting diagonal operator D by using a discrete version of the stabilization procedures developed in [15, 18, 16] (here, we just use combinatorial arguments instead of cluster points and randomization). Finally, by a perturbation argument, we obtain a constant multiple of the identity operator I_{Y_n}. In every reduction step, we obtain a factorization (or an approximate factorization) of the newly constructed operator through the previous one, and by combining all these factorizations, we obtain the desired results.

Observe that if the Haar system is K-unconditional in Y for some $K \geq 1$, then Theorem 1.2(ii) also holds for operators T for which the *absolute values* of all diagonal entries are at least δ (we just have to increase the factorization constant by a factor K). However, by performing an additional reduction step, we also can drop the assumption that T has positive diagonal entries in the general case. We have the following result:

COROLLARY 1.4. *Let Y be a Haar system Hardy space, and let $\Gamma, \delta, \varepsilon > 0$. Moreover, let $n, \widetilde{N} \in \mathbb{N}_0$ be chosen so that*

$$\widetilde{N} \geq 2N \left\lceil \frac{N}{\varepsilon} + 1 \right\rceil 2^N, \tag{1.3}$$

where $N = N_{\min}(2(1+\varepsilon)\Gamma, \delta, \varepsilon, n)$, or where $N = N_{\min}^K(2(1+\varepsilon)\Gamma, \delta, \varepsilon, n)$ if the Haar system is K-unconditional in Y. Then for every linear operator $\widetilde{T}\colon Y_{\widetilde{N}} \to Y_{\widetilde{N}}$ with $\|\widetilde{T}\| \leq \Gamma$ and with δ-large diagonal with respect to the Haar basis, the identity I_{Y_n} factors through $\widetilde{T}$ with constant $2(1+\varepsilon)^2/\delta$.

We will prove Corollary 1.4 in Section 5. Note that by (1.3), $\dim Y_{\widetilde{N}}$ is a double exponential function of $(\log(\dim Y_n))^2$, whereas in the unconditional case, $\dim Y_{\widetilde{N}}$ depends exponentially on $\dim Y_n$.

2. Preliminaries

2.1. Notation and basic definitions. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and let $\mathcal{D}$ denote the collection of all dyadic intervals in $[0,1)$, i.e.,

$$\mathcal{D} = \left\{ \left[\frac{i-1}{2^j}, \frac{i}{2^j} \right) : j \in \mathbb{N}_0,\ 1 \leq i \leq 2^j \right\}.$$

In addition, for each $n \in \mathbb{N}_0$, define

$$\mathcal{D}_n = \{ I \in \mathcal{D} : |I| = 2^{-n} \} \quad \text{and} \quad \mathcal{D}_{\leq n} = \bigcup_{k=0}^{n} \mathcal{D}_k, \quad \mathcal{D}_{<n} = \bigcup_{k=0}^{n-1} \mathcal{D}_k.$$

If $I \in \mathcal{D}$ is a dyadic interval, then we denote by I^+ the left half of I and by I^- the right half of I (both are elements of $\mathcal{D}$). If we use the symbol $\pm$ multiple times in an equation, we mean either always $+$ or always $-$. For any subcollection $\mathcal{B} \subset \mathcal{D}$, we put $\mathcal{B}^* = \bigcup_{I \in \mathcal{B}} I$.

Next, we define the bijective function $\iota \colon \mathcal{D} \to \mathbb{N}$ by

$$\left[\frac{i-1}{2^j}, \frac{i}{2^j}\right) \overset{\iota}{\mapsto} 2^j + i - 1.$$

The *Haar system* $(h_I)_{I \in \mathcal{D}}$ is defined as

$$h_I = \chi_{I^+} - \chi_{I^-}, \qquad I \in \mathcal{D},$$

where χ_A denotes the characteristic function of a subset $A \subset [0,1)$. We additionally put $h_\varnothing = \chi_{[0,1)}$ and $\mathcal{D}^+ = \mathcal{D} \cup \{\varnothing\}$ as well as $\iota(\varnothing) = 0$. Recall that in the linear order induced by ι, the Haar system $(h_I)_{I \in \mathcal{D}^+}$ is a monotone Schauder basis of L^p, $1 \le p < \infty$ (and unconditional if $1 < p < \infty$). For $x = \sum_{I \in \mathcal{D}^+} a_I h_I \in L^1$, by the *Haar support* of x we mean the set of all $I \in \mathcal{D}^+$ such that $a_I \neq 0$. We only consider Banach spaces over the real numbers. Finally, we will occasionally use the symbols $\gtrsim$, $\lesssim$, $\sim$ to suppress constants in inequalities: $a \gtrsim b$ and $b \lesssim a$ both mean that $a \ge Cb$ for an absolute constant $C > 0$, and $a \sim b$ means that $a \gtrsim b$ and $b \gtrsim a$.

2.2. Haar system Hardy spaces. The class of Haar system Hardy spaces was introduced in [18, 16]. To give the definition, we first need the notion of a Haar system space (see [15, Section 2.4]).

Definition 2.1. A *Haar system space* X is the completion of $H := \operatorname{span}\{h_I : I \in \mathcal{D}^+\} = \operatorname{span}\{\chi_I : I \in \mathcal{D}\}$ under a norm $\|\cdot\|_X$ that satisfies the following properties:

(i) If f, g are in H and $|f|, |g|$ have the same distribution, then $\|f\|_X = \|g\|_X$.
(ii) $\|\chi_{[0,1)}\|_X = 1$.

We denote the class of Haar system spaces by $\mathcal{H}(\delta)$. Given $X \in \mathcal{H}(\delta)$, one can define the closed subspace X_0 of X as the closure of $H_0 = \operatorname{span}\{h_I : I \in \mathcal{D}\}$ in X. We denote the class of these subspaces by $\mathcal{H}_0(\delta)$.

Apart from the spaces L^p, $1 \le p < \infty$, and the closure of H in L^∞, the class $\mathcal{H}(\delta)$ includes all rearrangement-invariant function spaces X on $[0,1)$ (e.g., Orlicz function spaces) in which the linear span H of the Haar system $(h_I)_{I \in \mathcal{D}^+}$ is dense. If H is not dense in X, then its closure in X is a Haar system space.

The following proposition contains some basic results on Haar system spaces which will be used frequently throughout the paper. For proofs we refer to [15, Section 2.4] and [18, Section 4].

Proposition 2.2. *Let $X \in \mathcal{H}(\delta)$ be a Haar system space. Then the following hold:*

(i) *For every $f \in H = \operatorname{span}\{h_I\}_{\in \mathcal{D}^+}$, we have $\|f\|_{L^1} \le \|f\|_X \le \|f\|_{L^\infty}$.*
(ii) *For all $f, g \in H$ with $|f| \le |g|$, we have $\|f\|_X \le \|g\|_X$.*
(iii) *The Haar system $(h_I)_{I\in\mathcal{D}^+}$, in the usual linear order, is a monotone Schauder basis of X.*
(iv) *H naturally coincides with a subspace of X^* (where $f \in H$ acts on $g \in H$ by $\langle f, g\rangle = \int fg$), and its closure in X^* is also a Haar system space.*
(v) *For every $I \in \mathcal{D}$, we have $\|h_I\|_X \|h_I\|_{X^*} = |I|$.*

Next, we denote by $(r_n)_{n=0}^\infty$ the sequence of standard Rademacher functions, i.e.,

$$r_n = \sum_{I\in\mathcal{D}_n} h_I, \qquad n \in \mathbb{N}_0.$$

We define the set

$$\mathcal{R} = \{(r_{\iota(I)})_{I\in\mathcal{D}^+}, (r_0)_{I\in\mathcal{D}^+}\}.$$

Hence, if $\mathbf{r} = (r_I)_{I\in\mathcal{D}^+} \in \mathcal{R}$, then $\mathbf{r}$ is either an independent sequence of ±1-valued random variables or a constant sequence.

Definition 2.3. Given $X \in \mathcal{H}(\delta)$ and $\mathbf{r} = (r_I)_{I\in\mathcal{D}^+} \in \mathcal{R}$, we define the (*one-parameter*) *Haar system Hardy space* $X(\mathbf{r})$ as the completion of $H = \operatorname{span}\{h_I : I \in \mathcal{D}^+\}$ under the norm $\|\cdot\|_{X(\mathbf{r})}$ given by

$$\Big\|\sum_{I\in\mathcal{D}^+} a_I h_I\Big\|_{X(\mathbf{r})} = \Big\|s \mapsto \int_0^1 \Big|\sum_{I\in\mathcal{D}^+} r_I(u) a_I h_I(s)\Big| \,\mathrm{d}u\Big\|_X.$$

We denote the class of one-parameter Haar system Hardy spaces by $\mathcal{HH}(\delta)$. Moreover, given $X(\mathbf{r}) \in \mathcal{HH}(\delta)$, we define the subspace $X_0(\mathbf{r})$ as the closure of $H_0 = \operatorname{span}\{h_I\}_{I\in\mathcal{D}}$ in $X(\mathbf{r})$. For notational convenience, we will also refer to the subspaces $X_0(\mathbf{r})$ as Haar system Hardy spaces. We denote the class of these subspaces by $\mathcal{HH}_0(\delta)$.

Note that if $r_I = r_0$ for all $I \in \mathcal{D}^+$, then $\|\cdot\|_{X(\mathbf{r})} = \|\cdot\|_X$ and thus $X(\mathbf{r}) = X$, so $\mathcal{HH}(\delta)$ is an extension of $\mathcal{H}(\delta)$. Moreover, if $\mathbf{r}$ is independent, then by Khinchin's inequality, the norm on $X(\mathbf{r})$ can be expressed in terms of the square function:

$$\Big\|\sum_{I\in\mathcal{D}^+} a_I h_I\Big\|_{X(\mathbf{r})} \sim \Big\|\Big(\sum_{I\in\mathcal{D}^+} a_I^2 h_I^2\Big)^{1/2}\Big\|_X.$$

Thus, for $X = L^1$ and an independent Rademacher sequence $\mathbf{r}$, we obtain the dyadic Hardy space H^1, equipped with an equivalent norm.

In the next proposition, we collect some basic properties of Haar system Hardy spaces.

PROPOSITION 2.4. *Let $X(\mathbf{r}) \in \mathcal{HH}(\delta)$, and put $Y = X_0(\mathbf{r})$. Then the following hold:*

(i) *The Haar system $(h_I)_{I\in\mathcal{D}}$, in the usual linear order, is a monotone Schauder basis of Y, and it is 1-unconditional if $\mathbf{r}$ is independent.*
(ii) *$H_0 = \operatorname{span}\{h_I\}_{I\in\mathcal{D}}$ naturally coincides with a subspace of Y^*, and if $\mathbf{r}$ is independent, then $(h_I)_{I\in\mathcal{D}}$ is 1-unconditional in Y^*.*
(iii) *For every $I \in \mathcal{D}$, we have $\|h_I\|_Y\|h_I\|_{Y^*} = |I|$. In particular, $\|h_I\|_Y = \|h_I\|_X$ and $\|h_I\|_{Y^*} = \|h_I\|_{X^*}$.*
(iv) *If f is a finite linear combination of disjointly supported Haar functions, then $\|f\|_{X(\mathbf{r})} = \|f\|_X$.*
(v) *If $x = \sum_{I\in\mathcal{D}} a_I h_I \in H_0$, then for every $k \in \mathbb{N}_0$ we have $\|\sum_{I\in\mathcal{D}_k} a_I h_I\|_Y \le \|x\|_Y$. Thus, $\|a_I h_I\|_Y \le \|x\|_Y$ for all $I \in \mathcal{D}$.*
(vi) *If $\mathcal{B} \subset \mathcal{D}$ is a finite collection of pairwise disjoint dyadic intervals, then $(h_K)_{K\in\mathcal{B}}$ is 1-unconditional in $X(\mathbf{r})^*$, and we have $\|\sum_{K\in\mathcal{B}} h_K\|_{X(\mathbf{r})^*} \le 1$.*

Proof. Most of the above results are either explicitly or implicitly contained in [18]. For completeness, we provide proofs or detailed references.

(i) If $\mathbf{r}$ is constant, i.e., $X(\mathbf{r}) = X$ and $Y = X_0$, then the statement holds by Proposition 2.2(iii). If $\mathbf{r}$ is independent, then $(h_I)_{I\in\mathcal{D}}$ is 1-unconditional in Y (this follows immediately from Definition 2.3), and in particular, it is a monotone Schauder basis of Y.

(ii) We identify $f \in H_0$ with the continuous extension of $g \mapsto \langle f, g\rangle = \int fg$, where $g \in H_0$. Then both statements follow from (i) since $(h_I/|I|)_{I\in\mathcal{D}^+}$ is the dual basic sequence in Y^* associated with the Haar basis $(h_I)_{I\in\mathcal{D}}$ of Y.

(iii) This is proved in [18, Lemma 4.4].

(iv) This is a direct consequence of Definition 2.3 and the fact that f consists of disjointly supported Haar functions.

(v) If $\mathbf{r}$ is independent, then both statements follow from the 1-unconditionality of the Haar system in $X(\mathbf{r})$. Otherwise, $X(\mathbf{r}) = X$ and $Y = X_0$. Now put $y = \sum_{I\in\mathcal{D}_{<k}} a_I h_I$ and $z = \sum_{I\in\mathcal{D}_k} a_I h_I$. Then $z + y$ and $z - y$ have the same distribution and their average is z. Thus,

$$\|z\|_X \le \tfrac{1}{2}(\|z+y\|_X + \|z-y\|_X) = \|y+z\|_X \le \|x\|_X,$$

where in the last step, we have used the fact that the Haar system is a monotone Schauder basis of Y. The additional statement follows since for every fixed k the system $(h_I)_{I\in\mathcal{D}_k}$ is 1-unconditional in Y.

(vi) If $\mathbf{r}$ is constant, then the 1-unconditionality follows from Proposition 2.2(iv) and Definition 2.1, and the inequality is a consequence of Proposition 2.2(ii), applied to the X^*-norm. If $\mathbf{r}$ is independent, then the Haar basis $(h_I)_{I\in\mathcal{D}^+}$ of $X(\mathbf{r})$ is 1-unconditional, so the same is true for its dual basic sequence $(h_I/|I|)_{I\in\mathcal{D}^+}$ in $X(\mathbf{r})^*$ and thus for $(h_K)_{K\in\mathcal{B}}$ in $X(\mathbf{r})^*$. To prove the last inequality, let $x = \sum_{I\in\mathcal{D}^+} a_I h_I \in H$ and observe that

$$\Big\langle \sum_{K\in\mathcal{B}} h_K, x\Big\rangle = \sum_{K\in\mathcal{B}} a_K|K| \le \Big\|\sum_{K\in\mathcal{B}} a_K h_K\Big\|_{L^1}$$
$$\le \Big\|\sum_{K\in\mathcal{B}} a_K h_K\Big\|_X = \Big\|\sum_{K\in\mathcal{B}} a_K h_K\Big\|_{X(\mathbf{r})} \le \|x\|_{X(\mathbf{r})},$$

where we have used Propositions 2.2(i) and 2.4(iv) and the 1-unconditionality of the Haar system in $X(\mathbf{r})$. ∎

In the following, Y will always denote a Haar system Hardy space $X_0(\mathbf{r}) \in \mathcal{HH}_0(\delta)$, and we will consider operators defined on the subspaces $Y_n = \operatorname{span}\{h_I\}_{I\in\mathcal{D}_{\le n}} \subset Y$, $n \in \mathbb{N}_0$.

2.3. Faithful Haar systems. Next, we introduce a finite version of the concept of a *faithful Haar system*. A faithful Haar system is a system of functions which are blocks of the Haar system and share many structural properties with the original Haar system. For example, the supports of these functions exhibit the same intersection pattern as the supports of the original Haar functions. The term *faithful Haar system* was coined in [15], but the construction can already be found, e.g., in [25, p. 51]. We will also use the more general notion of an *almost faithful Haar system*, which allows for gaps between the supports of the functions (cf. [18]). These types of constructions originated in classical works such as the paper [7] by J. L. B. Gamlen and R. J. Gaudet. For more information on these and related concepts, such as *Jones' compatibility conditions* [8], we refer to [21]. A brief overview can also be found in [18, Section 5].

Definition 2.5. Let $n \in \mathbb{N}_0$. For every $I \in \mathcal{D}_{\le n}$, let $\mathcal{B}_I$ be a non-empty finite subcollection of $\mathcal{D}$, and moreover, let $(\theta_K)_{K\in\mathcal{D}}$ be a family of signs. Put $\tilde{h}_I = \sum_{K\in\mathcal{B}_I} \theta_K h_K$ for $I \in \mathcal{D}_{\le n}$. We say that $(\tilde{h}_I)_{I\in\mathcal{D}_{\le n}}$ is a (finite) *almost faithful Haar system* if the following conditions are satisfied:

- (i) For every $I \in \mathcal{D}_{\le n}$, the collection $\mathcal{B}_I$ consists of pairwise disjoint dyadic intervals, and $\mathcal{B}_I \cap \mathcal{B}_J = \emptyset$ for all $I \ne J \in \mathcal{D}_{\le n}$.
- (ii) For every $I \in \mathcal{D}_{<n}$, we have $\mathcal{B}_{I\pm}^* \subset \{\tilde{h}_I = \pm 1\}$.

If $\mathcal{B}_{[0,1)}^* = [0,1)$ and $\mathcal{B}_{I\pm}^* = \{\tilde{h}_I = \pm 1\}$ for all $I \in \mathcal{D}_{<n}$, then we say that $(\tilde{h}_I)_{I\in\mathcal{D}_{\le n}}$ is *faithful*, and in this case, we usually denote the system by $(\hat{h}_I)_{I\in\mathcal{D}_{\le n}}$. If $k_0 < k_1 < \cdots < k_n$ is a strictly increasing sequence of integers such that $\mathcal{B}_I \subset \mathcal{D}_{k_i}$ for all $I \in \mathcal{D}_i$, $0 \le i \le n$, then we say that $(\tilde{h}_I)_{I\in\mathcal{D}_{\le n}}$ is an almost faithful Haar system *with frequencies* $k_0, \ldots, k_n$ (note that in the terminology of [18], the frequencies would be denoted as $(k_I)_{I\in\mathcal{D}_{\le n}}$, where

$k_I = k_i$ whenever $|I| = 2^{-i}$). An example of a (finite) faithful Haar system is depicted in Figure 1.

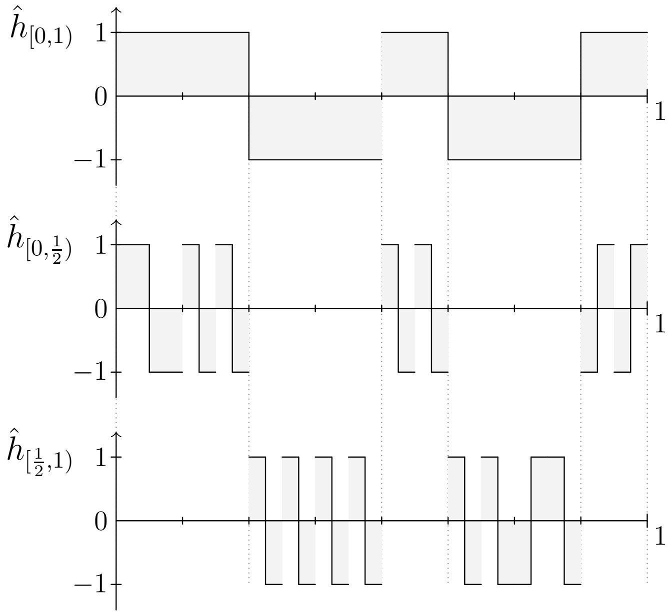

Fig. 1. A finite faithful Haar system

If $(\hat{h}_I)_{I\in\mathcal{D}_{\le n}}$ is a faithful Haar system and $N \in \mathbb{N}_0$ is chosen so that $\mathcal{B}_I \subset \mathcal{D}_{\le N}$ for all $I \in \mathcal{D}_{\le n}$, then we can always define two associated operators:

Proposition 2.6. *Let $Y \in \mathcal{HH}_0(\delta)$ be a Haar system Hardy space. Fix $n, N \in \mathbb{N}_0$, and let $(\hat{h}_I)_{I\in\mathcal{D}_{\le n}}$ be a (finite) faithful Haar system in Y_N. Define the associated operators $\hat{B}\colon Y_n \to Y_N$ and $\hat{A}\colon Y_N \to Y_n$ by*

$$\hat{B}x = \sum_{I\in\mathcal{D}_{\le n}} \frac{\langle h_I, x\rangle}{|I|}\hat{h}_I \quad \text{and} \quad \hat{A}y = \sum_{I\in\mathcal{D}_{\le n}} \frac{\langle \hat{h}_I, y\rangle}{|I|} h_I$$

for $x \in Y_n$ and $y \in Y_N$. Then $\hat{A}\hat{B} = I_{Y_n}$ and $\|\hat{A}\| = \|\hat{B}\| = 1$.

Proof. This result follows from [18, Proposition 7.1] by suitably extending $(\hat{h}_I)_{I\in\mathcal{D}_{\le n}}$ to an *infinite* faithful Haar system $(\hat{h}_I)_{I\in\mathcal{D}}$ (see [18, Definition 5.1]) and restricting the associated operators $\hat{B}$ and $\hat{A}$ to Y_n and Y_N, respectively. ■

In order to prove an analogous statement for *almost* faithful Haar systems, we will need an upper bound for the norm of a Haar multiplier, i.e., a linear operator $M\colon Y_n \to Y_n$ with $Mh_I = m_I h_I$ for all $I \in \mathcal{D}_{\le n}$, where

$(m_I)_{I\in\mathcal{D}_{\leq n}}$ is a family of scalars (the *entries* of M). In our finite-dimensional setting, the following inequality yields a better result than the bound in [18, Lemma 4.5].

Lemma 2.7. *Let Y be a Haar system Hardy space, and let $n \in \mathbb{N}_0$. Then for every Haar multiplier $M\colon Y_n \to Y_n$ with entries $(m_I)_{I\in\mathcal{D}_{\leq n}}$, we have*

$$\|M - m_{[0,1)}I_{Y_n}\| \leq \sum_{k=1}^{n} \max_{I\in\mathcal{D}_k} |m_I - m_{[0,1)}|.$$

Proof. Let $x = \sum_{I\in\mathcal{D}_{\leq n}} a_I h_I \in Y_n$. Then

$$\begin{aligned}\|Mx - m_{[0,1)}x\|_Y &= \Big\|\sum_{k=1}^{n}\sum_{I\in\mathcal{D}_k}(m_I - m_{[0,1)})a_I h_I\Big\|_Y\\ &\leq \sum_{k=1}^{n}\max_{I\in\mathcal{D}_k}|m_I - m_{[0,1)}|\Big\|\sum_{I\in\mathcal{D}_k} a_I h_I\Big\|_Y,\end{aligned}$$

and the conclusion follows since $\|\sum_{I\in\mathcal{D}_k} a_I h_I\|_Y \leq \|x\|_Y$ for all k by Proposition 2.4(v). ∎

Proposition 2.8. *Let Y be a Haar system Hardy space, and let $n, N \in \mathbb{N}_0$. Moreover, let $(\tilde{h}_I)_{I\in\mathcal{D}_{\leq n}}$ be an almost faithful Haar system in Y_N, and for each $I \in \mathcal{D}_{\leq n}$, let $\mathcal{B}_I$ denote the Haar support of $\tilde{h}_I$. Let $B\colon Y_n \to Y_N$ and $A\colon Y_N \to Y_n$ be the associated operators defined by*

$$Bx = \sum_{I\in\mathcal{D}_{\leq n}} \frac{\langle h_I, x\rangle}{|I|}\tilde{h}_I \quad \text{and} \quad Ay = \sum_{I\in\mathcal{D}_{\leq N}} \frac{\langle \tilde{h}_I, y\rangle}{|\mathcal{B}_I^*|} h_I$$

for $x \in Y_n$ and $y \in Y_N$. Put $\mu = |\mathcal{B}_{[0,1)}^*|$. Then $AB = I_{Y_n}$ and

$$\|B\| \leq 1 \quad \text{and} \quad \|A\| \leq \frac{1}{\mu} + \sum_{k=1}^{n} \max_{I\in\mathcal{D}_k}\Big(\frac{|I|}{|\mathcal{B}_I^*|} - \frac{1}{\mu}\Big).$$

Proof. Let $Q\colon Y_N \to Y_n$ be defined by

$$Qy = \sum_{I\in\mathcal{D}_{\leq N}} \frac{\langle \tilde{h}_I, y\rangle}{|I|} h_I, \qquad y \in Y_N.$$

As in the proof of [18, Theorem 7.3], we see that the linear operators B and Q satisfy $\|B\| \leq 1$ and $\|Q\| \leq 1$. Note that $A = MQ$, where $M\colon Y_n \to Y_n$ is the Haar multiplier defined by

$$Mh_I = \frac{|I|}{|\mathcal{B}_I^*|}h_I, \qquad I \in \mathcal{D}_{\leq n}.$$

Thus, the conclusion follows from Lemma 2.7. ∎

2.4. Factorization of operators. Finally, we introduce some convenient terminology for stating factorization results and describing properties of linear operators and their diagonals (cf. [15, 18]).

Definition 2.9. Let X and Y denote Banach spaces. Let $S\colon X \to X$ and $T\colon Y \to Y$ be bounded linear operators, and let $C, \eta \geq 0$.

(i) We say that S *factors through* T *with constant* C *and error* η if there exist bounded linear operators $B\colon X \to Y$ and $A\colon Y \to X$ with $\|A\|\,\|B\| \leq C$ such that $\|S - ATB\| \leq \eta$.
(ii) If (i) holds and additionally $AB = I_X$, then we say that S *projectionally* factors through T with constant C and error η.

If we omit the phrase "with error η" in (i) or (ii), then we take that to mean that the error is 0.

The following observation will play an important role in the proof of Theorem 1.2(i): Let X, Y and S, T be as above and suppose that S *projectionally* factors through T with constant C and error η. Then $I_X - S$ projectionally factors through $I_Y - T$ with the same constant and error (cf. [15, Remark 2.2]).

Definition 2.10. Let Y be a Haar sytem Hardy space. Moreover, let $\delta > 0$, let $n \in \mathbb{N}_0$, and let $T\colon Y_n \to Y_n$ be a linear operator. We say that

▷ T has *δ-large diagonal* (*with respect to the Haar system*) if $|\langle h_I, Th_I\rangle| \geq \delta|I|$ for all $I \in \mathcal{D}_{\leq n}$;
▷ T has δ-large *positive* diagonal if $\langle h_I, Th_I\rangle \geq \delta|I|$ for all $I \in \mathcal{D}_{\leq n}$.

3. Diagonalization via random faithful Haar systems. As a first step towards proving our main results, we reduce an arbitrary linear operator $T\colon Y_N \to Y_N$ to a Haar multiplier $D\colon Y_n \to Y_n$, where N depends linearly on n. This is achieved by constructing a randomized faithful Haar system $(\hat{h}_I)_{I\in\mathcal{D}_{\leq n}}$ in Y_N and using the associated operators $\hat{A}, \hat{B}$ as defined in Proposition 2.6: By a probabilistic argument based on [12], we show that for at least one realization of our random system, we have an approximate factorization $D = \hat{A}T\hat{B} + \varDelta$, where D is a Haar multiplier and $\varDelta$ is a small error. Moreover, if T has δ-large positive diagonal with respect to the Haar system for some $\delta > 0$, then the diagonalization preserves this property (the new diagonal entries will only be slightly smaller).

In [12], R. Lechner used the following randomized construction to prove a diagonalization result in the case where Y is the dyadic Hardy space H^p, $1 \leq p < \infty$, or SL^∞. The first step is to select pairwise disjoint finite collections $\mathcal{B}_I \subset \mathcal{D}_{\leq N}$ which determine a faithful Haar system

$$b_I = \sum_{K\in\mathcal{B}_I} h_K, \qquad I \in \mathcal{D}_{\leq n}.$$

Next, random signs $\theta = (\theta_K)_{K\in\mathcal{D}_{\le N}}$ are introduced to obtain the functions

$$b_I^{(\theta)} = \sum_{K\in\mathcal{B}_I} \theta_K h_K, \qquad I \in \mathcal{D}_{\le n},\, \theta \in \{\pm 1\}^{\mathcal{D}_{\le N}}.$$

Note that in general $(b_I^{(\theta)})_{I\in\mathcal{D}_{\le n}}$ is *not* a faithful Haar system since the newly introduced signs θ_K can break property (ii) of Definition 2.5. At this point, the proof in [12] exploits the fact that the Haar system $(h_K)_{K\in\mathcal{D}}$ is 1-unconditional in H^p and SL^∞: For each θ, the function $b_I^{(\theta)}$ is obtained by applying a fixed Haar multiplier of norm 1 to b_I. Thus, the associated operators $\hat{A}$ and $\hat{B}$, defined as in Proposition 2.6 with respect to $(b_I^{(\theta)})_{I\in\mathcal{D}_{\le n}}$, are also bounded, and for at least one choice of θ they yield the desired approximate factorization $D = \hat{A}T\hat{B} + \Delta$.

Recently, a refinement of this probabilistic technique was applied by K. Konstantos and P. Motakis [9] to prove factorization results in the Bourgain–Rosenthal–Schechtman spaces R^p_ω, $1 < p < \infty$. In our case, however, Y is an arbitrary Haar system Hardy space, so in general we cannot use unconditionality to complete the proof as described above. However, a modified randomization procedure, which always yields a *faithful* Haar system, enables us to prove the diagonalization result. Finally, it is worth noting that our resulting Haar multiplier D is already stable along every level of the Haar system, i.e., if $I, J \in \mathcal{D}_{\le n}$ satisfy $|I| = |J|$, then the corresponding entries of D satisfy $d_I \approx d_J$ (we will utilize this property later).

Before stating and proving our diagonalization result, Proposition 3.2, we introduce the probabilistic framework that will be used to construct a randomized faithful Haar system which almost diagonalizes a given operator T.

Fix $m, n \in \mathbb{N}_0$, put $M = 2^m$, and let $N \ge m + n$. Let $\theta = (\theta_K)_{K\in\mathcal{D}_{\le N}}$ be chosen uniformly at random from $\{\pm 1\}^{\mathcal{D}_{\le N}}$. We denote the uniform measure on $\{\pm 1\}^{\mathcal{D}_{\le N}}$ by $\mathbb{P}$ and the corresponding expected value and variance by $\mathbb{E}$ and $\mathbb{V}$, respectively. Moreover, for $0 \le k \le n$ and $1 \le i \le M$, let

$$\theta_k = (\theta_K : K \in \mathcal{D}_{m+k}) \quad \text{and} \quad \theta_k^i = (\theta_K : K \in \mathcal{D}_{m+k},\, K \subset F_i),$$

where $(F_i)_{1\le i\le M}$ is an enumeration of the dyadic intervals in $\mathcal{D}_m$. Hence, we start with the entire family of signs θ, extract $n+1$ levels and split them according to the partition $(F_i)_{1\le i\le M}$ of $[0, 1)$. Finally, let $\mathbb{E}_k$ and $\mathbb{E}_k^i$ denote the conditional expectations with respect to the σ-algebras generated by $(\theta_K : K \notin \mathcal{D}_{m+k})$ and $(\theta_K : K \notin \mathcal{D}_{m+k} \vee K \not\subset F_i)$, respectively. Thus, applying $\mathbb{E}_k$ (or $\mathbb{E}_k^i$) to a real-valued function of θ is the same as averaging over all realizations of θ_k (or θ_k^i) while leaving all the other random variables θ_K, which do not appear in θ_k (or θ_k^i), fixed.

Now we can inductively construct a randomized faithful Haar system $(\hat{h}_I(\theta))_{I\in\mathcal{D}_{\le n}}$ with frequencies $m, m+1, \dots, m+n$ by putting $\mathcal{B}_{[0,1)}(\theta) = \mathcal{D}_m$

and

$$\hat{h}_I(\theta) = \sum_{K\in\mathcal{B}_I(\theta)} \theta_K h_K, \qquad I \in \mathcal{D}_{\leq n}, \tag{3.1}$$

where

$$\mathcal{B}_{I^\pm}(\theta) = \{K \in \mathcal{D}_{m+k+1} : K \subset \{\hat{h}_I(\theta) = \pm 1\}\}, \quad I \in \mathcal{D}_k,\ k = 0, 1, \dots, n-1.$$

This is illustrated in Figure 2 for two different choices of θ (the signs θ_K are displayed directly above the corresponding Haar functions h_K).

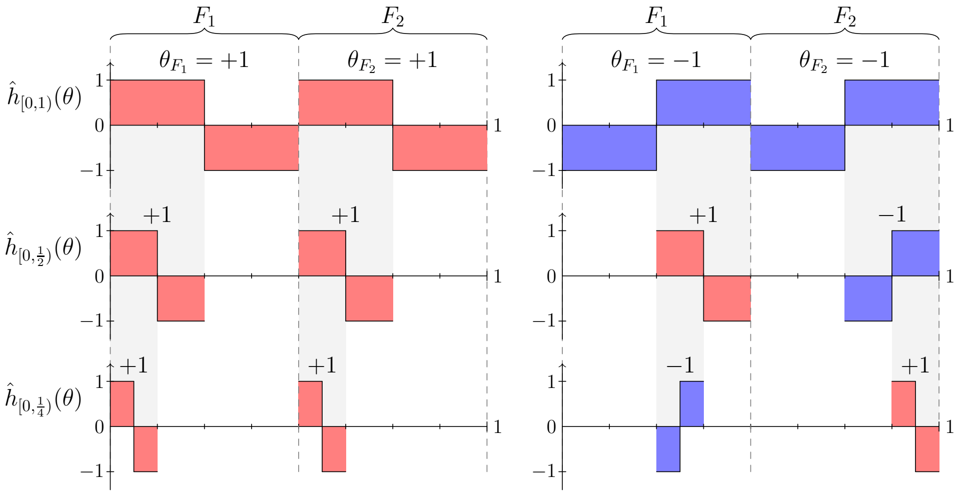

Fig. 2. Two realizations of a randomized faithful Haar system for $m = 1$. The constructions below F_1 and F_2 are independent—this is crucial for the discussion following (3.6).

Note that if $I \in \mathcal{D}_k$ for some $0 \leq k \leq n$, then the collection $\mathcal{B}_I(\theta)$ only depends on the signs in $\theta_0, \theta_1, \dots, \theta_{k-1}$, whereas the function $\hat{h}_I(\theta)$ also depends on the signs in θ_k since they appear as factors in (3.1). Thus, we may write

$$\mathcal{B}_I(\theta) = \mathcal{B}_I(\theta_0, \dots, \theta_{k-1}) \quad \text{and} \quad \hat{h}_I(\theta) = \hat{h}_I(\theta_0, \dots, \theta_k). \tag{3.2}$$

We can also go one step further: By induction on k, we see that for every $I \in \mathcal{D}_k$ $(0 \leq k \leq n)$ and every $1 \leq i \leq M$, the collection

$$\mathcal{B}_I^i(\theta) := \{K \in \mathcal{B}_I(\theta) : K \subset F_i\}$$

contains exactly one element, and hence, the restriction $\hat{h}_I(\theta) \cdot \chi_{F_i}$ is a single Haar function, multiplied with a random sign. Moreover, note that $\mathcal{B}_I^i(\theta)$ only depends on the values of those signs θ_K for which $K \subset F_i$. For example, we have $\mathcal{B}^i_{[0,1)}(\theta) = \{F_i\}$ and $\mathcal{B}^i_{[0,1/2)}(\theta) = \{F_i^+\}$ if $\theta_{F_i} = +1$, whereas

$\mathcal{B}^i_{[0,1/2)}(\theta) = \{F_i^-\}$ if $\theta_{F_i} = -1$. Thus, we can write

$$\mathcal{B}^i_I(\theta) = \mathcal{B}^i_I(\theta^i_0, \dots, \theta^i_{k-1})$$

for all $1 \le i \le M$. In fact, as we let θ vary, the unique element of $\mathcal{B}^i_I(\theta)$ ranges uniformly over all $K \in \mathcal{D}_{m+k}$ with $K \subset F_i$.

Next, we prove our main probabilistic lemma, which controls the diagonal and off-diagonal terms of a given operator T with respect to a randomized faithful Haar system.

LEMMA 3.1. *Let $m, n \in \mathbb{N}_0$, let $N \ge m + n$, and let $(\hat{h}_I(\theta))_{I \in \mathcal{D}_{\le n}}$ be a randomized faithful Haar system with frequencies $m, m+1, \dots, m+n$ defined as above. Moreover, let Y be a Haar system Hardy space, and let $T\colon Y_N \to Y_N$ be a linear operator. Then the random variables*

$$X_{I,J}(\theta) = \langle \hat{h}_I(\theta), T\hat{h}_J(\theta)\rangle, \qquad I, J \in \mathcal{D}_{\le n},$$

satisfy

$$\mathbb{E}X_{I,J} = 0 \quad \text{and} \quad \mathbb{V}X_{I,J} \le \|T\|^2 2^{-m/2} \qquad \text{if } I \ne J$$

as well as

$$\mathbb{E}X_{I,I} = |I| \cdot 2^{-m}|I| \sum_{\substack{K \in \mathcal{D}\\ |K| = 2^{-m}|I|}} \frac{\langle h_K, Th_K\rangle}{|K|} \quad \text{and} \quad \mathbb{V}X_{I,I} \le 3\|T\|^2 2^{-m/2}.$$

Proof. Let $k, l \in \{0, \dots, n\}$ and fix dyadic intervals $I \in \mathcal{D}_k$ and $J \in \mathcal{D}_l$. Then

$$X_{I,J}(\theta) = \sum_{\substack{K \in \mathcal{B}_I(\theta)\\ L \in \mathcal{B}_J(\theta)}} \theta_K \theta_L \langle h_K, Th_L\rangle,$$

$$X_{I,J}(\theta)^2 = \sum_{\substack{K, K' \in \mathcal{B}_I(\theta)\\ L, L' \in \mathcal{B}_J(\theta)}} \theta_K \theta_{K'} \theta_L \theta_{L'} \langle h_K, Th_L\rangle \langle h_{K'}, Th_{L'}\rangle.$$

Expected value of $X_{I,J}$. We assume that $k \ge l$ (the case $k < l$ is analogous). Since by (3.2), the collections $\mathcal{B}_I(\theta)$ and $\mathcal{B}_J(\theta)$ do not depend on any of the signs in θ_k, we have

$$\mathbb{E}X_{I,J} = \mathbb{E}(\mathbb{E}_k X_{I,J}) = \mathbb{E} \sum_{\substack{K \in \mathcal{B}_I(\theta_0, \dots, \theta_{k-1})\\ L \in \mathcal{B}_J(\theta_0, \dots, \theta_{l-1})}} \mathbb{E}_k(\theta_K \theta_L) \langle h_K, Th_L\rangle, \tag{3.3}$$

i.e., the conditional expectation $\mathbb{E}_k$ commutes with the double sum. If $I \ne J$, then (3.3) yields $\mathbb{E}X_{I,J} = 0$ since the collections $\mathcal{B}_I(\theta_0, \dots, \theta_{k-1})$ and $\mathcal{B}_J(\theta_0, \dots, \theta_{l-1})$ are disjoint for each realization of $\theta_0, \dots, \theta_{k-1}$, which implies that the random variables θ_K and θ_L appearing in (3.3) are independent. On the other hand, if $I = J$, then

$$\mathbb{E}X_{I,I} = \mathbb{E} \sum_{K,L\in\mathcal{B}_I(\theta_0,\dots,\theta_{k-1})} \mathbb{E}_k(\theta_K\theta_L)\langle h_K, Th_L\rangle \tag{3.4}$$
$$= \mathbb{E} \sum_{K\in\mathcal{B}_I(\theta_0,\dots,\theta_{k-1})} \langle h_K, Th_K\rangle = \sum_{i=1}^{M} \mathbb{E} \sum_{K\in\mathcal{B}_I^i(\theta)} \langle h_K, Th_K\rangle$$
$$= \sum_{i=1}^{M} \frac{1}{2^k} \sum_{\substack{K\in\mathcal{D}_{m+k}\\ K\subset F_i}} \langle h_K, Th_K\rangle = \frac{1}{2^k} \sum_{K\in\mathcal{D}_{m+k}} \langle h_K, Th_K\rangle$$
$$= |I|\cdot 2^{-m}|I| \sum_{\substack{K\in\mathcal{D}\\ |K|=2^{-m}|I|}} \frac{\langle h_K, Th_K\rangle}{|K|}.$$

Variance of $X_{I,J}$ for $I \neq J$. First, we assume that $I \neq J$ and $k = l$, i.e., $|I| = |J|$. Exploiting the facts that $\mathbb{E} = \mathbb{E}\mathbb{E}_k$, that the collections $\mathcal{B}_I(\theta) = \mathcal{B}_I(\theta_0,\dots,\theta_{k-1})$ and $\mathcal{B}_J(\theta) = \mathcal{B}_J(\theta_0,\dots,\theta_{k-1})$ are disjoint, and that the components of θ_k are independent random variables, we obtain

$$\mathbb{E}X_{I,J}^2 = \mathbb{E} \sum_{\substack{K,K'\in\mathcal{B}_I(\theta_0,\dots,\theta_{k-1})\\ L,L'\in\mathcal{B}_J(\theta_0,\dots,\theta_{k-1})}} \underbrace{\mathbb{E}_k(\theta_K\theta_{K'}\theta_L\theta_{L'})}_{=\mathbb{E}_k(\theta_K\theta_{K'})\mathbb{E}_k(\theta_L\theta_{L'})} \langle h_K, Th_L\rangle\langle h_{K'}, Th_{L'}\rangle \tag{3.5}$$
$$= \mathbb{E} \sum_{\substack{K\in\mathcal{B}_I(\theta)\\ L\in\mathcal{B}_J(\theta)}} \langle h_K, Th_L\rangle^2.$$

Next, let $I \neq J$ and $k > l$. Then, as above, we can use $\mathbb{E} = \mathbb{E}\mathbb{E}_k$ and exchange $\mathbb{E}_k$ with the sum over K, K', L, L' to obtain

$$\mathbb{E}X_{I,J}^2 = \mathbb{E} \sum_{\substack{K,K'\in\mathcal{B}_I(\theta_0,\dots,\theta_{k-1})\\ L,L'\in\mathcal{B}_J(\theta_0,\dots,\theta_{l-1})}} \mathbb{E}_k(\theta_K\theta_{K'})\,\theta_L\theta_{L'}\langle h_K, Th_L\rangle\langle h_{K'}, Th_{L'}\rangle$$
$$= \mathbb{E} \sum_{\substack{K\in\mathcal{B}_I(\theta)\\ L,L'\in\mathcal{B}_J(\theta)}} \theta_L\theta_{L'}\langle h_K, Th_L\rangle\langle h_K, Th_{L'}\rangle,$$

where we have also used the fact that θ_K and $\theta_{K'}$ are independent if $K \neq K'$. We can rewrite the last equation in terms of the collections $\mathcal{B}_I^i(\theta)$ as follows:

$$\mathbb{E}X_{I,J}^2 = \sum_{i,j,j'=1}^{M} \mathbb{E} \sum_{\substack{K\in\mathcal{B}_I^i(\theta),\\ L\in\mathcal{B}_J^j(\theta),\, L'\in\mathcal{B}_J^{j'}(\theta)}} \theta_L\theta_{L'}\langle h_K, Th_L\rangle\langle h_K, Th_{L'}\rangle. \tag{3.6}$$

Now consider fixed indices i, j, j'. If $j \neq j'$, then $j \neq i$ or $j' \neq i$. Suppose that $j \neq i$, and replace $\mathbb{E}$ by $\mathbb{E}\mathbb{E}_l^j$ in (3.6). Recall that we can write

$\mathcal{B}_I^i(\theta) = \mathcal{B}_I^i(\theta_0^i, \dots, \theta_{k-1}^i)$ as well as $\mathcal{B}_J^j(\theta) = \mathcal{B}_J^j(\theta_0^j, \dots, \theta_{l-1}^j)$ and $\mathcal{B}_J^{j'}(\theta) = \mathcal{B}_J^{j'}(\theta_0^{j'}, \dots, \theta_{l-1}^{j'})$, so none of these collections depends on the signs in θ_l^j since $j \neq i$ and $l > l-1$. Hence, the conditional expectation $\mathbb{E}_l^j$ commutes with the inner triple sum in (3.6). Moreover, the summands satisfy

$$\mathbb{E}_l^j(\theta_L \theta_{L'})\langle h_K, Th_L\rangle\langle h_K, Th_{L'}\rangle = \mathbb{E}_l^j(\theta_L)\theta_{L'}\langle h_K, Th_L\rangle\langle h_K, Th_{L'}\rangle = 0,$$

since $\theta_{L'}$ is not a component of θ_l^j for $L' \in \mathcal{B}_J^{j'}(\theta)$ as $j \neq j'$. If $j' \neq i$, then the same reasoning as above shows that $\mathbb{E}_l^{j'}(\theta_L\theta_{L'}) = 0$. Thus, we may restrict the first sum in (3.6) to the case $j = j'$, and since $\mathcal{B}_J^j(\theta)$ is a singleton, we can also put $L = L'$ and $\theta_L\theta_{L'} = 1$ in (3.6). Hence, we obtain

$$\mathbb{E}X_{I,J}^2 = \mathbb{E}\sum_{i,j=1}^{M} \sum_{\substack{K \in \mathcal{B}_I^i(\theta) \\ L \in \mathcal{B}_J^j(\theta)}} \langle h_K, Th_L\rangle^2 = \mathbb{E}\sum_{\substack{K \in \mathcal{B}_I(\theta) \\ L \in \mathcal{B}_J(\theta)}} \langle h_K, Th_L\rangle^2. \tag{3.7}$$

If $k < l$, then (3.7) holds by analogous arguments, and for $k = l$, we have already proved it above (see (3.5)).

Following [12, proof of Theorem 4.1], we will now estimate (3.7) in two different ways. We fix θ, put $a_{K,L} = \langle h_K, Th_L\rangle$ for $K, L \in \mathcal{D}_{\leq N}$ and note that

$$|a_{K,L}| \leq \|h_K\|_{Y^*}\|T\|\,\|h_L\|_Y.$$

By the 1-unconditionality of $(h_L)_{L\in\mathcal{B}_J(\theta)}$ in Y, we have

$$\begin{aligned}\sum_{\substack{K \in \mathcal{B}_I(\theta) \\ L \in \mathcal{B}_J(\theta)}} \langle h_K, Th_L\rangle^2 &= \sum_{K\in\mathcal{B}_I(\theta)} \Big\langle h_K, T\Big(\sum_{L\in\mathcal{B}_J(\theta)} a_{K,L}h_L\Big)\Big\rangle \\ &\leq \sum_{K\in\mathcal{B}_I(\theta)} \|h_K\|_{Y^*}\|T\| \max_{L\in\mathcal{B}_J(\theta)} |a_{K,L}| \Big\|\sum_{L\in\mathcal{B}_J(\theta)} h_L\Big\|_Y \\ &\leq \|T\|^2 \sum_{K\in\mathcal{B}_I(\theta)} \|h_K\|_{Y^*}^2 \max_{L\in\mathcal{B}_J(\theta)} \|h_L\|_Y,\end{aligned}$$

where the last inequality follows from Proposition 2.4(iv) combined with Proposition 2.2(ii) and the fact that $\|\chi_{[0,1)}\|_X = 1$ in every Haar system space X. Now we use the inequalities

$$\|h_L\|_Y \leq \|h_{F_1}\|_Y, \quad \|h_K\|_{Y^*}^2 = \|h_K\|_{Y^*}\frac{|K|}{\|h_K\|_Y} \leq \|h_{F_1}\|_{Y^*}\frac{|K|}{|I|\|h_{F_1}\|_Y} \tag{3.8}$$

for $K \in \mathcal{B}_I(\theta)$ and $L \in \mathcal{B}_J(\theta)$, which can proved as follows: Replace $\|\cdot\|_Y$ and $\|\cdot\|_{Y^*}$ with the underlying Haar system space norm $\|\cdot\|_X$ and its dual norm $\|\cdot\|_{X^*}$, respectively, using Proposition 2.4(iii). Then apply Proposition 2.2(ii), (iv) as well as the first equation in Proposition 2.4(iii). The inequality $\|h_K\|_X \geq |I|\,\|h_{F_1}\|_X$ for $|K| = |I|\,|F_1|$ is obtained by repeatedly applying the triangle inequality and the rearrangement-invariance of $\|\cdot\|_X$. Using (3.8) and $|\mathcal{B}_I(\theta)^*| = |I|$ (which is true for every faithful Haar system), we conclude that

$$\sum_{\substack{K\in\mathcal{B}_I(\theta)\\ L\in\mathcal{B}_J(\theta)}} \langle h_K, Th_L\rangle^2 \le \|T\|^2 \|h_{F_1}\|_{Y^*} \sum_{K\in\mathcal{B}_I(\theta)} \frac{|K|}{|I|} = \|T\|^2\|h_{F_1}\|_{Y^*}. \tag{3.9}$$

On the other hand, we can rewrite (3.7) using

$$\sum_{\substack{K\in\mathcal{B}_I(\theta)\\ L\in\mathcal{B}_J(\theta)}} \langle h_K, Th_L\rangle^2 = \sum_{L\in\mathcal{B}_J(\theta)} \Big\langle \sum_{K\in\mathcal{B}_I(\theta)} a_{K,L} h_K, Th_L\Big\rangle.$$

Then we perform an analogous computation to the one above, using Proposition 2.4(vi) applied to the collection $\mathcal{B}_I(\theta)$ (note that $\|\cdot\|_{Y^*} \le \|\cdot\|_{X(\mathbf{r})^*}$) as well as suitable versions of the estimates (3.8). In the end, we obtain

$$\sum_{\substack{K\in\mathcal{B}_I(\theta)\\ L\in\mathcal{B}_J(\theta)}} \langle h_K, Th_L\rangle^2 \le \|T\|^2 \|h_{F_1}\|_{Y} \sum_{L\in\mathcal{B}_J(\theta)} \frac{|L|}{|J|} = \|T\|^2\|h_{F_1}\|_{Y}. \tag{3.10}$$

Combining equation (3.7) with the estimates (3.9) and (3.10) yields

$$\begin{aligned} \mathbb{V}X_{I,J} = \mathbb{E}X_{I,J}^2 &\le \|T\|^2\sqrt{\|h_{F_1}\|_Y\|h_{F_1}\|_{Y^*}} \\ &= \|T\|^2\sqrt{|F_1|} = \|T\|^2 2^{-m/2}, \end{aligned} \tag{3.11}$$

where we again used Proposition 2.4(iii).

Variance of $X_{I,I}$. As above, we exploit the facts that $\mathbb{E} = \mathbb{E}\mathbb{E}_k$ and that $\mathbb{E}_k$ commutes with the sum over K, K', L, L' to obtain

$$\begin{aligned} \mathbb{E}X_{I,I}^2 &= \mathbb{E} \sum_{K,K',L,L'\in\mathcal{B}_I(\theta)} \theta_K\theta_{K'}\theta_L\theta_{L'}\langle h_K, Th_L\rangle\langle h_{K'}, Th_{L'}\rangle \\ &= \mathbb{E} \sum_{K,K',L,L'\in\mathcal{B}_I(\theta_0,\dots,\theta_{k-1})} \mathbb{E}_k(\theta_K\theta_{K'}\theta_L\theta_{L'})\,\langle h_K, Th_L\rangle\langle h_{K'}, Th_{L'}\rangle. \end{aligned}$$

Now observe that for $K, K', L, L' \in \mathcal{B}_I(\theta)$, we have

$$\mathbb{E}_k(\theta_K\theta_{K'}\theta_L\theta_{L'}) = 1$$

if one of the following conditions (i)–(iv) holds, and otherwise, by independence, the expression is 0 (see [12, proof of Theorem 4.1]):

(i) $K = L = K' = L'$,
(ii) $K = L \ne K' = L'$,
(iii) $K = K' \ne L = L'$,
(iv) $K = L' \ne L = K'$.

Hence, we can write $\mathbb{E}X_{I,I}^2 = V_1 + V_2 + V_3 + V_4$, where

$$V_1 = \mathbb{E}\sum_{K\in\mathcal{B}_I(\theta)} \langle h_K, Th_K\rangle^2, \qquad V_2 = \mathbb{E}\sum_{\substack{K,L\in\mathcal{B}_I(\theta)\\ K\neq L}} \langle h_K, Th_K\rangle\langle h_L, Th_L\rangle,$$

$$V_3 = \mathbb{E}\sum_{\substack{K,L\in\mathcal{B}_I(\theta)\\ K\neq L}} \langle h_K, Th_L\rangle^2, \quad V_4 = \mathbb{E}\sum_{\substack{K,L\in\mathcal{B}_I(\theta)\\ K\neq L}} \langle h_K, Th_L\rangle\langle h_L, Th_K\rangle.$$

Observe that

$$\begin{aligned} V_2 &= \mathbb{E}\sum_{\substack{i,j=1\\ i\neq j}}^{M}\Big(\sum_{K\in\mathcal{B}_I^i(\theta)} \langle h_K, Th_K\rangle\Big)\Big(\sum_{L\in\mathcal{B}_I^j(\theta)} \langle h_L, Th_L\rangle\Big)\\ &= \sum_{\substack{i,j=1\\ i\neq j}}^{M}\Big(\mathbb{E}\sum_{K\in\mathcal{B}_I^i(\theta)} \langle h_K, Th_K\rangle\Big)\Big(\mathbb{E}\sum_{L\in\mathcal{B}_I^j(\theta)} \langle h_L, Th_L\rangle\Big), \end{aligned}$$

since $\mathcal{B}_I^i(\theta) = \mathcal{B}_I^i(\theta_0^i,\ldots,\theta_{k-1}^i)$ and $\mathcal{B}_I^j(\theta) = \mathcal{B}_I^j(\theta_0^j,\ldots,\theta_{k-1}^j)$ are independent for $i\neq j$. Together with (3.4), this implies that

$$(\mathbb{E}X_{I,I})^2 = \Big(\sum_{i=1}^{M}\mathbb{E}\sum_{K\in\mathcal{B}_I^i(\theta)} \langle h_K, Th_K\rangle\Big)^2 \geq V_2,$$

and thus

$$\mathbb{V}X_{I,I} \leq V_1 + V_3 + V_4. \tag{3.12}$$

Hence, it suffices to give upper bounds for V_1, V_3 and V_4. By Proposition 2.4(iii), and since $|\mathcal{B}_I(\theta)^*| = |I|$ and $|K| = |I|2^{-m}$ for all $K\in\mathcal{B}_I(\theta)$, we obtain

$$V_1 \leq \mathbb{E}\sum_{K\in\mathcal{B}_I(\theta)} \|h_K\|_{Y^*}^2\|T\|^2\|h_K\|_Y^2 = \|T\|^2\mathbb{E}\sum_{K\in\mathcal{B}_I(\theta)} |K|^2 = \|T\|^2|I|^2 2^{-m}.$$

Moreover, the double sums in V_3 and V_4 can be estimated using the same techniques as in the computation of $\mathbb{V}X_{I,J}$ for $I\neq J$, yielding

$$V_3 \leq \|T\|^2 2^{-m/2} \quad\text{and}\quad V_4 \leq \|T\|^2 2^{-m/2}.$$

By plugging these estimates into (3.12), we obtain

$$\mathbb{V}X_{I,I} \leq 3\|T\|^2 2^{-m/2}. \ \blacksquare$$

We can now prove our main diagonalization result.

Proposition 3.2. *Let $Y = X_0(\mathbf{r})$ be a Haar system Hardy space, and let $\Gamma,\delta,\eta > 0$. Moreover, let $n, N\in\mathbb{N}_0$ be chosen so that*

$$N \geq 21(n+1) + \Big\lfloor 4\log_2\Big(\frac{\Gamma}{\eta\delta}\Big)\Big\rfloor.$$

Then for every linear operator $T\colon Y_N \to Y_N$ with $\|T\| \le \Gamma$, there exists a Haar multiplier $D\colon Y_n \to Y_n$ with entries $(d_I)_{I\in\mathcal{D}_{\le n}}$ such that D projectionally factors through T with constant 1 and error $\eta\delta$. Moreover, the following hold:

(i) $|d_I| \le \|T\|$ *for all* $I \in \mathcal{D}_{\le n}$.
(ii) $|d_I - d_J| \le 8^{-n}\eta\delta$ *whenever* $I, J \in \mathcal{D}_{\le n}$ *satisfy* $|I| = |J|$.
(iii) *If T has δ-large positive diagonal, then D has $(1-\eta)\delta$-large positive diagonal.*

Proof. Fix $n \in \mathbb{N}_0$, and let $m \in \mathbb{N}_0$ be the smallest integer for which

$$2^m > \frac{2^{4(n+2)}\Gamma^4}{\eta_0^4}, \quad \text{where} \quad \eta_0 = \frac{\eta\delta}{2^{4n+2}}. \tag{3.13}$$

Put $M = 2^m$ and fix an integer $N \ge n + m$, i.e.,

$$N \ge n+1+\left\lfloor 4(n+2)+4\log_2\left(\frac{\Gamma}{\eta\delta}\right)+4(4n+2)\right\rfloor = 21n+17+\left\lfloor 4\log_2\left(\frac{\Gamma}{\eta\delta}\right)\right\rfloor.$$

Moreover, let $T\colon Y_N \to Y_N$ be a linear operator with $\|T\| \le \Gamma$. Let θ be chosen uniformly at random from $\{\pm1\}^{\mathcal{D}_{\le N}}$, and let $(\hat{h}_I(\theta))_{I\in\mathcal{D}_{\le n}}$ be the randomized faithful Haar system with frequencies $m, m+1, \ldots, m+n$ introduced just before Lemma 3.1. Now define $X_{I,J}(\theta) = \langle \hat{h}_I(\theta), T\hat{h}_J(\theta)\rangle$ for $I, J \in \mathcal{D}_{\le n}$. As in [12], we consider the off-diagonal events

$$O_{I,J} = \{\theta \in \{\pm1\}^{\mathcal{D}_{\le N}} : |X_{I,J}(\theta)| \ge \eta_0\}, \qquad I, J \in \mathcal{D}_{\le n},\ I \ne J$$

and the diagonal events

$$D_I = \{\theta \in \{\pm1\}^{\mathcal{D}_{\le N}} : |X_{I,I}(\theta) - \mathbb{E}X_{I,I}| \ge \eta_0\}, \qquad I \in \mathcal{D}_{\le n}.$$

Using Chebyshev's inequality and our upper bounds for $\mathbb{V}X_{I,J}$ and $\mathbb{V}X_{I,I}$ from Lemma 3.1, we obtain

$$\mathbb{P}(O_{I,J}) \le \frac{\Gamma^2}{2^{m/2}\eta_0^2} \quad \text{and} \quad \mathbb{P}(D_I) \le \frac{3\Gamma^2}{2^{m/2}\eta_0^2}, \qquad I, J \in \mathcal{D}_{\le n},\ I \ne J.$$

Hence, by (3.13), we have

$$\begin{aligned}\mathbb{P}\Big(\bigcup_{\substack{I,J\in\mathcal{D}_{\le n}\\ I\ne J}} O_{I,J} \cup \bigcup_{I\in\mathcal{D}_{\le n}} D_I\Big) &\le \sum_{\substack{I,J\in\mathcal{D}_{\le n}\\ I\ne J}} \mathbb{P}(O_{I,J}) + \sum_{I\in\mathcal{D}_{\le n}} \mathbb{P}(D_I)\\ &\le \frac{2^{2(n+2)}\Gamma^2}{2^{m/2}\eta_0^2} < 1.\end{aligned}$$

Thus, we can find at least one family of signs $\theta \in \{\pm1\}^{\mathcal{D}_{\le N}}$ such that the corresponding faithful Haar system $(\hat{h}_I)_{I\in\mathcal{D}_{\leq n}} = (\hat{h}_I(\theta))_{I\in\mathcal{D}_{\leq n}}$ satisfies

$$\frac{|\langle \hat{h}_I, T\hat{h}_J\rangle|}{|I|} \leq \frac{\eta_0}{|I|} \leq 2^n\eta_0, \quad I,J\in\mathcal{D}_{\leq n},\ I\neq J, \tag{3.14}$$

$$\left|\frac{\langle \hat{h}_I, T\hat{h}_I\rangle}{|I|} - 2^{-m}|I| \sum_{\substack{K\in\mathcal{D}\\ |K|=2^{-m}|I|}} \frac{\langle h_K, Th_K\rangle}{|K|}\right| \leq \frac{\eta_0}{|I|} \leq 2^n\eta_0, \quad I\in\mathcal{D}_{\leq n}, \tag{3.15}$$

where we also inserted the expected value $\mathbb{E}X_{I,I}$ computed in Lemma 3.1.

Now let $\hat{A}\colon Y_N\to Y_n$ and $\hat{B}\colon Y_n\to Y_N$ denote the operators associated with $(\hat{h}_I)_{I\in\mathcal{D}_{\leq n}}$ as defined in Proposition 2.6. Put

$$d_I = \frac{\langle \hat{h}_I, T\hat{h}_I\rangle}{|I|}, \qquad I\in\mathcal{D}_{\leq n},$$

and consider the Haar multiplier $D\colon Y_n\to Y_n$ defined as the linear extension of $Dh_I = d_I h_I$, $I\in\mathcal{D}_{\leq n}$. We have $|d_I|\leq\|T\|$ for all $I\in\mathcal{D}_{\leq n}$ (this follows, e.g., from Propositions 2.6 and 2.4(v)). Observe that by (3.15) and $2^{n+1}\eta_0\leq 8^{-n}\eta\delta$ (cf. (3.13)), we have $|d_I-d_J|\leq 8^{-n}\eta\delta$ whenever $|I|=|J|$. Moreover, if T has δ-large positive diagonal, i.e., $\langle h_K, Th_K\rangle\geq\delta|K|$ for all $K\in\mathcal{D}_{\leq N}$, then (3.15) together with $2^n\eta_0\leq\eta\delta$ implies that $d_I\geq(1-\eta)\delta$ for all $I\in\mathcal{D}_{\leq n}$.

Finally, let $x=\sum_{J\in\mathcal{D}_{\leq n}} a_J h_J\in Y_n$. Then, using (3.14) and the inequality $|a_J|\leq\|x\|_Y/\|h_J\|_Y\leq 2^n\|x\|_Y$ for $J\in\mathcal{D}_{\leq n}$ (see Proposition 2.4(v)), we obtain

$$\begin{aligned}\|(\hat{A}T\hat{B}-D)x\|_Y &= \Big\|\sum_{J\in\mathcal{D}_{\leq n}}\sum_{\substack{I\in\mathcal{D}_{\leq n}\\ I\neq J}} a_J\frac{\langle\hat{h}_I, T\hat{h}_J\rangle}{|I|}h_I\Big\|_Y\\ &\leq 2^n\eta_0\sum_{J\in\mathcal{D}_{\leq n}}\sum_{\substack{I\in\mathcal{D}_{\leq n}\\ I\neq J}}|a_J|\,\|h_I\|_Y\leq 2^{4n+2}\eta_0\|x\|_Y,\end{aligned}$$

and hence, by (3.13),

$$\|\hat{A}T\hat{B}-D\|\leq 2^{4n+2}\eta_0\leq\eta\delta. \ \blacksquare$$

4. Stabilization of Haar multipliers. Our next goal is to reduce the Haar multiplier D obtained from Proposition 3.2 to a constant multiple of the identity. This is achieved by first *stabilizing* D and then employing a perturbation argument. The infinite-dimensional stabilization methods from [15, 18, 16] involving cluster points and probabilistic methods will be replaced by a finite, combinatorial version which just uses the pigeonhole principle. Note that in the stabilization part, we will also make use of the fact that the entries of D are already stable along every level of the dyadic tree (see Proposition 3.2(ii)).

PROPOSITION 4.1. *Let Y be a Haar system Hardy space, and let $\Gamma, \delta, \eta > 0$. Moreover, let $n, N \in \mathbb{N}_0$ be chosen so that*

$$N \geq 42n(n+1)\left\lceil \frac{\Gamma}{\eta\delta} \right\rceil + 42 + \left\lfloor 4\log_2\left(\frac{\Gamma}{\eta\delta}\right) \right\rfloor. \tag{4.1}$$

Then for every linear operator $T\colon Y_N \to Y_N$ with $\|T\| \leq \Gamma$, there exists a scalar c with $|c| \leq \|T\|$ such that cI_{Y_n} projectionally factors through T with constant 1 and error $3\eta\delta$. Moreover, if T has δ-large positive diagonal, then $c \geq (1-\eta)\delta$. If the Haar system is K-unconditional in Y for some $K \geq 1$, then inequality (4.1) can be replaced by

$$N \geq 42n\left\lceil \frac{K\Gamma}{\eta\delta} \right\rceil + 42 + \left\lfloor 4\log_2\left(\frac{\Gamma}{\eta\delta}\right) \right\rfloor.$$

Proof. Let $n \in \mathbb{N}_0$ and fix $\Gamma, \delta, \eta > 0$. Put

$$\tilde{n} = 2n(n+1)\left\lceil \frac{\Gamma}{\eta\delta} \right\rceil + 1 \geq n\left\lceil \frac{2\Gamma(n+1)}{\eta\delta} \right\rceil + 1 \tag{4.2}$$

and let $N \in \mathbb{N}_0$ satisfy (4.1), i.e., $N \geq 21(\tilde{n}+1) + \left\lfloor 4\log_2\left(\frac{\Gamma}{\eta\delta}\right) \right\rfloor$. Let $T\colon Y_N \to Y_N$ be a linear operator with $\|T\| \leq \Gamma$. By Proposition 3.2, there exists a Haar multiplier $D\colon Y_{\tilde{n}} \to Y_{\tilde{n}}$ with entries $(d_K)_{K\in\mathcal{D}_{\leq\tilde{n}}}$ such that D projectionally factors through T with constant 1 and error $\eta\delta$, and such that $|d_K| \leq \|T\| \leq \Gamma$ for all $K \in \mathcal{D}_{\leq\tilde{n}}$ and

$$|d_K - d_L| \leq 8^{-\tilde{n}}\eta\delta \leq 8^{-n}\eta\delta \quad \text{whenever } |K| = |L|. \tag{4.3}$$

Moreover, if T has δ-large positive diagonal, then $d_I \geq (1-\eta)\delta$ for all $I \in \mathcal{D}_{\leq\tilde{n}}$.

Now consider the entries $d_{[0,2^{-k})}$, $0 \leq k \leq \tilde{n}$. Divide the interval $[-\Gamma, \Gamma]$ into $\lceil 2\Gamma(n+1)/(\eta\delta) \rceil$ subintervals of length at most $\eta\delta/(n+1)$. Then, by (4.2) and the pigeonhole principle, we can find natural numbers $0 \leq k_0 < k_1 < \cdots < k_n \leq \tilde{n}$ such that

$$|d_{[0,2^{-k_i})} - d_{[0,2^{-k_0})}| \leq \frac{\eta\delta}{n+1}, \qquad i = 0, \ldots, n. \tag{4.4}$$

Hence, after putting $c = d_{[0,2^{-k_0})}$, we have $|c| \leq \|T\|$ and, by (4.3) and (4.4),

$$|d_K - c| \leq \frac{2\eta\delta}{n+1}, \qquad K \in \mathcal{D}_{k_0} \cup \cdots \cup \mathcal{D}_{k_n}. \tag{4.5}$$

If T has δ-large positive diagonal, then $c \geq (1-\eta)\delta$. Now let $(\hat{h}_I)_{I\in\mathcal{D}_{\leq n}}$ be a faithful Haar system with frequencies $k_0, \ldots, k_n$, i.e., for every $I \in \mathcal{D}_i$ $(0 \leq i \leq n)$, we have $\mathcal{B}_I \subset \mathcal{D}_{k_i}$, where $\mathcal{B}_I$ is the Haar support of $\hat{h}_I$. Let $\hat{A}\colon Y_{\tilde{n}} \to Y_n$ and $\hat{B}\colon Y_n \to Y_{\tilde{n}}$ denote the associated operators as defined in Proposition 2.6. Then $D^{\mathrm{stab}} := \hat{A}D\hat{B}$ is also a Haar multiplier, and its entries $(d_I^{\mathrm{stab}})_{I\in\mathcal{D}_{\le n}}$ satisfy

$$d_I^{\mathrm{stab}} = \frac{\langle \hat{h}_I, D\hat{h}_I\rangle}{|I|} = \sum_{K\in\mathcal{B}_I} d_K \frac{|K|}{|I|}, \qquad I \in \mathcal{D}_{\le n}.$$

Together with (4.5), this implies that

$$|d_I^{\mathrm{stab}} - c| \le \frac{2\eta\delta}{n+1}, \qquad I\in\mathcal{D}_{\le n}. \tag{4.6}$$

We will now show that $\|D^{\mathrm{stab}} - cI_{Y_n}\| \le 2\eta\delta$. To this end, let $x = \sum_{I\in\mathcal{D}_{\le n}} a_I h_I \in Y_n$. Then, exploiting the 1-unconditionality of $(h_I)_{I\in\mathcal{D}_k}$ in Y for each $k\in\mathbb{N}_0$ and using Proposition 2.4(v), we obtain

$$\begin{aligned}\|(D^{\mathrm{stab}} - cI_{Y_n})x\|_Y &= \Big\|\sum_{I\in\mathcal{D}_{\le n}} (d_I^{\mathrm{stab}} - c)a_I h_I\Big\|_Y \\ &\le \sum_{k=0}^{n}\Big\|\sum_{I\in\mathcal{D}_k}(d_I^{\mathrm{stab}} - c)a_I h_I\Big\|_Y \le \frac{2\eta\delta}{n+1}\sum_{k=0}^{n}\Big\|\sum_{I\in\mathcal{D}_k} a_I h_I\Big\|_Y \\ &\le \frac{2\eta\delta}{n+1}\sum_{k=0}^{n}\|x\|_Y = 2\eta\delta\|x\|_Y,\end{aligned}$$

i.e., $\|D^{\mathrm{stab}} - cI_{Y_n}\| \le 2\eta\delta$. Recall that D projectionally factors through T with constant 1 and error $\eta\delta$, and D^{stab} projectionally factors through D with constant 1 and error 0. By combining all these statements, we deduce that cI_{Y_n} projectionally factors through T with constant 1 and error $3\eta\delta$.

If the Haar system is K-unconditional in Y, then we replace (4.2) by

$$\tilde{n} = 2n\left\lceil\frac{K\Gamma}{\eta\delta}\right\rceil + 1 \ge n\left\lceil\frac{2K\Gamma}{\eta\delta}\right\rceil + 1,$$

and we also replace the factor $n+1$ by K in the pigeonhole argument and in the denominator in (4.6). This inequality then directly implies $\|D^{\mathrm{stab}} - cI_{Y_n}\| \le 2\eta\delta$, and the result follows as above. ∎

Remark 4.2. If the space under consideration is L^1, then, of course, the results by Bourgain and Tzafriri [4] imply factorization results with linear dimension dependence (see Section 1). However, our method can also be modified to yield a better result in L^1 in the sense that inequality (4.1) is replaced by a bound of the form $N \ge C(\Gamma,\delta,\eta)n$: Replace the definition of $\tilde{n}$ in the proof of Proposition 4.1 by $\tilde{n} = l\cdot n$ for some $l\ge 1$ to be determined later, and let $D\colon Y_{\tilde{n}} \to Y_{\tilde{n}}$ be the Haar multiplier obtained from Proposition 3.2. Write $d_k = d_{[0,2^{-k})}$ for $k = 0,\dots,\tilde{n}$. Moreover, let $\bar{D}\colon Y_{\tilde{n}}\to Y_{\tilde{n}}$ be the Haar multiplier defined by $\bar{D}h_I = d_k h_I$ for all $I\in\mathcal{D}_k$, $0\le k\le\tilde{n}$. By (4.3) and the triangle inequality, we have $\|D-\bar{D}\| \le 2\eta\delta$. Next, we use a result by Semenov and Uksusov [26] (see also [30, 15] for alternative proofs), which characterizes the bounded Haar multipliers $M\colon L^1 \to L^1$ by identifying a quantity $|||M|||$ that is equivalent to the operator norm of M: In the formulation of [15, Theorem 2.6], we have

$$\|M\| \sim |||M||| := \sup_{(I_k)} \Big(\sum_{k=0}^{\infty} |m_{I_k} - m_{I_{k+1}}| + \limsup_{k\to\infty} |m_{I_k}| \Big),$$

where the supremum is taken over all sequences $(I_k)_{k=0}^{\infty}$ in $\mathcal{D}^+$ such that $I_0 = \varnothing$, $I_1 = [0,1)$, and $I_{k+1} \in \{I_k^+, I_k^-\}$ for every $k \geq 1$. A straightforward argument shows that we can restrict the domain to $Y_{\tilde{n}}$ and apply this result to $\bar{D}$ to obtain

$$\|\bar{D}\| \sim |d_0 - d_1| + \cdots + |d_{\tilde{n}-1} - d_{\tilde{n}}| + |d_{\tilde{n}}|. \tag{4.7}$$

Since $\|\bar{D}\| \leq \Gamma + 3\eta\delta$, we can find $0 \leq s \leq \tilde{n} - n$ such that

$$\frac{\Gamma + 3\eta\delta}{l} \gtrsim |d_s - d_{s+1}| + \cdots + |d_{s+n-1} - d_{s+n}|.$$

By constructing a faithful Haar system $(\hat{h}_I)_{I\in\mathcal{D}_{\leq n}}$ with frequencies $s, s+1, \ldots, s+n$ and using the associated operators $\hat{A}, \hat{B}$, we obtain a Haar multiplier $D^{\mathrm{stab}} = \hat{A}\bar{D}\hat{B}$, which satisfies $D^{\mathrm{stab}} h_I = d_{s+k} h_I$ for all $I \in \mathcal{D}_k$, $0 \leq k \leq n$. Therefore, applying (4.7) to D^{stab} and choosing, for example, $l \geq \frac{\Gamma}{3\eta\delta} + 1$ yields $\|D^{\mathrm{stab}} - d_{s+n} I_{Y_n}\| \lesssim \eta\delta$.

Despite the fact that the dimension dependence can be improved in the unconditional case and also in the case of L^1 (and, by duality, in L^∞), we do not know the answer to the following question:

QUESTION 4.3. *Is it true that for every Haar system Hardy space Y, an inequality of the form $N \geq C(\Gamma, \delta, \eta, Y)n$ is sufficient for the conclusion of Proposition* 4.1 (*and hence Theorem* 1.2) *to hold?*

Using the stabilization result Proposition 4.1, we are now able to prove our main result.

Proof of Theorem 1.2. We first prove (ii). Suppose that the linear operator $T\colon Y_N \to Y_N$ with $\|T\| \leq \Gamma$ has δ-large positive diagonal with respect to the Haar system. Then by Proposition 4.1, we can find a scalar $c \geq (1-\eta)\delta$ and operators $A\colon Y_N \to Y_n$ and $B\colon Y_n \to Y_N$ such that $\|A\|\,\|B\| \leq 1$ and $\|cI_{Y_n} - ATB\| \leq 3\eta\delta$. Thus,

$$\Big\| I_{Y_n} - \frac{1}{c} ATB \Big\| \leq \frac{3\eta\delta}{(1-\eta)\delta} = \frac{3\eta}{1-\eta} < 1.$$

This implies that the operator $Q = \frac{1}{c} ATB$ is invertible, and its inverse satisfies

$$\|Q^{-1}\| \leq \frac{1}{1 - 3\eta/(1-\eta)}.$$

Hence, we have the factorization $I_{Y_n} = \frac{1}{c}ATBQ^{-1}$, where

$$\left\|\frac{1}{c}A\right\|\|BQ^{-1}\| \leq \frac{\|Q^{-1}\|}{c} \leq \frac{1}{\delta(1-4\eta)} \leq \frac{1+\varepsilon}{\delta}.$$

To prove (i), we assume that $\delta = 1$. By Proposition 4.1, we can find a scalar c with $|c| \leq \|T\|$ such that cI_{Y_n} *projectionally* factors through T with constant 1 and error 3η. Thus, we also know that $(1-c)I_{Y_n}$ factors through $I_{Y_N} - T$ with the same constant and error. Note that we have either $c \geq 1/2$ or $1 - c \geq 1/2$. In the latter case, we replace T by $I_{Y_N} - T$ and c by $1 - c$. Then the proof can be completed in the same manner as above, with slightly modified estimates and an additional factor 2 in the factorization constant.

The results for spaces in which the Haar system is unconditional follow from the last statement in Proposition 4.1. ■

5. Reduction to positive diagonal. Finally, we consider the case of an operator $\widetilde{T}\colon Y_{\widetilde{N}} \to Y_{\widetilde{N}}$ with δ-large, not necessarily positive diagonal. If the Haar system is K-unconditional in Y, then by composing $\widetilde{T}$ with a Haar multiplier with entries ± 1, we obtain an operator $T\colon Y_{\widetilde{N}} \to Y_{\widetilde{N}}$ with δ-large *positive* diagonal such that T factors through $\widetilde{T}$ with constant K. Then Theorem 1.2 can be applied to T.

In the general case, however, a more sophisticated construction is needed to reduce $\widetilde{T}$ to an operator $T\colon Y_N \to Y_N$ with positive diagonal. We employ a discrete version of the *Gamlen–Gaudet construction.* This technique goes back to [7] (see also [21]), and the infinite version was also utilized, for example, in [14, 18]. Due to the combinatorial arguments used in this method, we need to assume an inequality of the form $\widetilde{N} \geq C(\varepsilon)N^2 2^N$.

PROPOSITION 5.1. *Let Y be a Haar system Hardy space, and let $\varepsilon, \delta > 0$. Moreover, let $N, \widetilde{N} \in \mathbb{N}_0$ be chosen so that*

$$\widetilde{N} \geq 2N\left\lceil \frac{N}{\varepsilon} + 1 \right\rceil 2^N. \tag{5.1}$$

Then for every linear operator $\widetilde{T}\colon Y_{\widetilde{N}} \to Y_{\widetilde{N}}$ with δ-large diagonal, there exists a linear operator $T\colon Y_N \to Y_N$ with δ-large positive *diagonal such that T factors through $\widetilde{T}$ with constant $2(1+\varepsilon)$.*

Proof. Suppose that $\widetilde{N} \geq 2lN$ for some $l \in \mathbb{N}$ to be determined later, and let $\widetilde{T}\colon Y_{\widetilde{N}} \to Y_{\widetilde{N}}$ be a linear operator with δ-large diagonal (with respect to the Haar basis). Put

$$\mathcal{A}_1 = \{K \in \mathcal{D}_{\leq \widetilde{N}} : \langle h_K, \widetilde{T}h_K\rangle \geq \delta|K|\},$$
$$\mathcal{A}_2 = \{K \in \mathcal{D}_{\leq \widetilde{N}} : \langle h_K, \widetilde{T}h_K\rangle \leq -\delta|K|\}.$$

Thus, $\mathcal{A}_1$ and $\mathcal{A}_2$ form a partition of $\mathcal{D}_{\leq \widetilde{N}}$. Now, for $i = 1, 2$ and $K \in \mathcal{D}_{\widetilde{N}}$, let

$$\mathcal{A}_i(K) = \{L \in \mathcal{A}_i : K \subset L\}.$$

Observe that for every $K \in \mathcal{D}_{\widetilde{N}}$, either $\mathcal{A}_1(K)$ or $\mathcal{A}_2(K)$ contains strictly more than $\widetilde{N}/2 \geq lN$ elements. Hence, if we define

$$\mathcal{C}_i = \{K \in \mathcal{D}_{\widetilde{N}} : \#\mathcal{A}_i(K) > lN\}, \quad i = 1, 2$$

(where # denotes cardinality), then $\mathcal{C}_1^* \cup \mathcal{C}_2^* = [0, 1)$, which implies that either $|\mathcal{C}_1^*| \geq 1/2$ or $|\mathcal{C}_2^*| \geq 1/2$. We may assume without loss of generality that $|\mathcal{C}_1^*| \geq 1/2$ (otherwise, we replace $\widetilde{T}$ by $-\widetilde{T}$).

Now, for a given collection of dyadic intervals $\mathcal{A} \subset \mathcal{D}$, we inductively define the generations

$$\mathcal{G}_0(\mathcal{A}) = \{K \in \mathcal{A} : K \text{ is maximal with respect to inclusion}\},$$
$$\mathcal{G}_k(\mathcal{A}) = \mathcal{G}_0\Big(\mathcal{A} \setminus \bigcup_{j=0}^{k-1} \mathcal{G}_j(\mathcal{A})\Big), \quad k \geq 1.$$

We will use the abbreviation $\mathcal{G}_k = \mathcal{G}_k(\mathcal{A}_1)$, $k \geq 0$. Thus, $\mathcal{G}_0, \mathcal{G}_1, \ldots$ are pairwise disjoint collections of pairwise disjoint dyadic intervals, and for every k, the collection $\mathcal{G}_k$ consists of those intervals $K \in \mathcal{A}_1$ which are strictly contained in exactly k intervals of $\mathcal{A}_1$. In particular, $\mathcal{G}_0^* \supset \mathcal{G}_1^* \supset \cdots$, and if $k \geq 1$, then for every $K \in \mathcal{G}_k$, there exists an interval $L \in \mathcal{G}_{k-1}$ such that $K \subset L^+$ or $K \subset L^-$. See Figure 3 for an example.

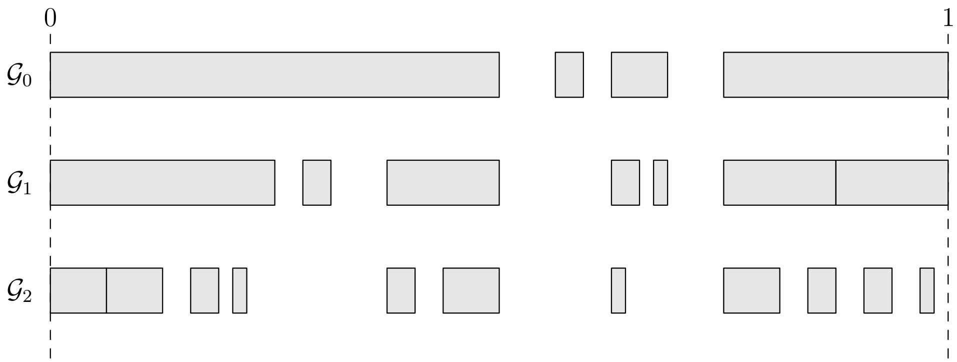

Fig. 3. The collections $\mathcal{G}_k$

Moreover, the definition of $\mathcal{C}_1$ and the above characterization of the collections $\mathcal{G}_k$ imply that for every $K \in \mathcal{C}_1$ and $0 \leq k \leq lN$, there is an $L \in \mathcal{A}_1(K) \cap \mathcal{G}_k$, and in particular $\mathcal{G}_{lN}^* \supset \mathcal{C}_1^*$. As a consequence, we have

$$|\mathcal{G}_{lN}^*| \geq |\mathcal{C}_1^*| \geq \frac{1}{2}.$$

Together with $|\mathcal{G}_0^*| \leq 1$, this yields

$$\frac{|\mathcal{G}_{lN}^*|}{|\mathcal{G}_{(l-1)N}^*|} \cdots \frac{|\mathcal{G}_{2N}^*|}{|\mathcal{G}_N^*|} \cdot \frac{|\mathcal{G}_N^*|}{|\mathcal{G}_0^*|} \geq \frac{1}{2},$$

and so there exists some $0 \leq s \leq (l-1)N$ such that $|\mathcal{G}_{s+N}^*| \geq 2^{-1/l}|\mathcal{G}_s^*|$. We choose

$$l \geq \left(\frac{N}{\varepsilon} + 1\right)2^N \ln 2 \geq -\frac{1}{\log_2\bigl(1 - \frac{1}{(N/\varepsilon+1)2^N}\bigr)},$$

where the last inequality follows from the standard inequality $\ln(1-x) \leq -x$ for $x < 1$ by plugging in $x = \frac{1}{(N/\varepsilon+1)2^N}$ and rearranging. Thus, we obtain

$$|\mathcal{G}_{s+N}^*| \geq \left(1 - \frac{1}{(N/\varepsilon + 1)2^N}\right)|\mathcal{G}_s^*|. \tag{5.2}$$

Next, we construct an almost faithful Haar system $(\tilde{h}_I)_{I \in \mathcal{D}_{\leq N}}$ by alternately choosing collections $\mathcal{B}_I \subset \mathcal{A}_1$ and signs $(\theta_K)_{K \in \mathcal{B}_I}$: First, we put $\mathcal{B}_{[0,1)} = \mathcal{G}_s$. If $\mathcal{B}_I \subset \mathcal{A}_1$ has already been constructed for some $I \in \mathcal{D}_k$, $0 \leq k \leq N$, then we choose the signs $(\theta_K)_{K \in \mathcal{B}_I}$ uniformly at random from $\{\pm 1\}^{\mathcal{B}_I}$. Since $\mathbb{E}(\theta_K\theta_L) = 1$ if and only if $K = L$ (and $\mathbb{E}(\theta_K\theta_L) = 0$ otherwise), the resulting function $\tilde{h}_I = \sum_{K \in \mathcal{B}_I} \theta_K h_K$ satisfies

$$\begin{aligned}\mathbb{E}\langle \tilde{h}_I, \widetilde{T}\tilde{h}_I\rangle &= \sum_{K,L \in \mathcal{B}_I} \mathbb{E}(\theta_K\theta_L)\langle h_K, \widetilde{T}h_L\rangle = \sum_{K \in \mathcal{B}_I} \langle h_K, \widetilde{T}h_K\rangle \\ &\geq \sum_{K \in \mathcal{B}_I} \delta|K| = \delta|\mathcal{B}_I^*|.\end{aligned}$$

Thus, there is at least one realization $(\theta_K)_{K \in \mathcal{B}_I}$ such that

$$\langle \tilde{h}_I, \widetilde{T}\tilde{h}_I\rangle \geq \delta|\mathcal{B}_I^*|. \tag{5.3}$$

We choose such a realization to define $\tilde{h}_I$. If $k < N$, then the successors $\mathcal{B}_{I\pm}$ are given by

$$\mathcal{B}_{I\pm} = \{K \in \mathcal{G}_{s+k+1} : K \subset \{\tilde{h}_I = \pm 1\}\},$$

and the construction continues.

Now let $A\colon Y_{\widetilde{N}} \to Y_N$ and $B\colon Y_N \to Y_{\widetilde{N}}$ be the operators associated with the resulting system $(\tilde{h}_I)_{I \in \mathcal{D}_{\leq N}}$. We will use Proposition 2.8 to estimate $\|A\|$ and $\|B\|$. Put $\mu = |\mathcal{B}_{[0,1)}^*| = |\mathcal{G}_s^*| \geq 1/2$. Note that for $1 \leq k \leq N$ and $I \in \mathcal{D}_k$ we have $|\mathcal{B}_I^*| \leq |I|\mu$, and thus

$$\max_{I \in \mathcal{D}_k}(|I|\mu - |\mathcal{B}_I^*|) \leq \sum_{I \in \mathcal{D}_k}(|I|\mu - |\mathcal{B}_I^*|) = |\mathcal{G}_s^*| - |\mathcal{G}_{s+k}^*| \leq \frac{\mu}{(N/\varepsilon + 1)2^N},$$

where the last inequality follows from (5.2). Hence,

$$|\mathcal{B}_I^*| \geq \left(1 - \frac{1}{N/\varepsilon + 1}\right)|I|\mu, \qquad I \in \mathcal{D}_{\leq N},$$

and this implies that

$$0 \le \frac{|I|}{|\mathcal{B}_I^*|} - \frac{1}{\mu} \le \frac{\varepsilon}{N\mu}, \qquad I \in \mathcal{D}_{\le N}.$$

Thus, by Proposition 2.8, we have $\|B\| \le 1$ and $\|A\| \le 2(1+\varepsilon)$. Now put $T = A\widetilde{T}B$. Then by (5.3), T has δ-large *positive* diagonal. ∎

Proof of Corollary 1.4. By combining Proposition 5.1 with Theorem 1.2, we obtain the claimed result (note that the operator T obtained from Proposition 5.1 satisfies $\|T\| \le 2(1+\varepsilon)\|\widetilde{T}\|$). ∎

We conclude by posing the following question:

QUESTION 5.2. *Does the conclusion of Corollary* 1.4 *still hold if the right-hand side of inequality* (1.3) *is replaced by a subexponential* (*e.g., polynomial*) *function of* N*?*

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Thomas Speckhofer
Institute of Analysis
Johannes Kepler University Linz
A-4040 Linz, Austria
E-mail: thomas.speckhofer@jku.at