**Problem 12163.** *Proposed by Thomas Speckhofer, Attnang-Puchheim, Austria.* Let $\mathbb{R}^n$ have the usual dot product and norm. When $v = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $\Sigma v = x_1 + \cdots + x_n$. Prove

$$\|v\|^2\|w\|^2 \geq (v \cdot w)^2 + \frac{1}{n}(\|v\||\Sigma w| - \|w\||\Sigma v|)^2$$

for all $v, w \in \mathbb{R}^n$.

**Suggested solution** (Thomas Speckhofer). A direct computation shows that if $(V, \langle\cdot,\cdot\rangle)$ is an inner product space and $v_1, \ldots, v_m \in V$, then the Gramian matrix $(\langle v_i, v_j\rangle)_{i,j=1}^m \in \mathbb{R}^{m\times m}$ is positive semidefinite. Hence, for all $u, v, w \in \mathbb{R}^n$, we have

$$\begin{vmatrix} u\cdot u & u\cdot v & u\cdot w \\ v\cdot u & v\cdot v & v\cdot w \\ w\cdot u & w\cdot v & w\cdot w \end{vmatrix} \geq 0$$

or, equivalently,

$$(u\cdot u)(v\cdot v)(w\cdot w) + 2(u\cdot v)(v\cdot w)(w\cdot u) \geq (u\cdot u)(v\cdot w)^2 + (v\cdot v)(u\cdot w)^2 + (w\cdot w)(u\cdot v)^2.$$

Plugging in $u = (1, \ldots, 1) \in \mathbb{R}^n$ yields

$$n\|v\|^2\|w\|^2 + 2(v\cdot w)(\Sigma v)(\Sigma w) \geq n(v\cdot w)^2 + \|v\|^2(\Sigma w)^2 + \|w\|^2(\Sigma v)^2.$$

Thus, we conclude that

$$\begin{aligned} n\big(\|v\|^2\|w\|^2 - (v\cdot w)^2\big) &\geq (\|v\|\Sigma w)^2 + (\|w\|\Sigma v)^2 - 2(v\cdot w)(\Sigma v)(\Sigma w) \\ &\geq (\|v\|\Sigma w)^2 + (\|w\|\Sigma v)^2 - 2|v\cdot w||\Sigma v||\Sigma w|. \end{aligned} \tag{1}$$

By applying the Cauchy-Schwarz inequality $|v\cdot w| \leq \|v\|\|w\|$ on the right-hand side of (1), we obtain

$$n\big(\|v\|^2\|w\|^2 - (v\cdot w)^2\big) \geq (\|v\||\Sigma w| - \|w\||\Sigma w|)^2,$$

which gives the desired result. $\square$